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Ex. 25 *Prove that* $\log_3 5$ *is an irrational.*

$\log_3 5 = \frac{p}{q}$ ← wrong

$5 = 3^{p/q}$

$5^q = 3^p$ ×

Ex. 30 *Given,* $a^2 + b^2 = c^2$. *Prove that*

$\log_{b+c} a + \log_{c-b} a = 2\log_{c+b} a \cdot \log_{c-b} a, \forall a > 0, a \neq 1$

$c - b > 0, c + b > 0$

$c - b \neq 1, c + b \neq 1.$

$\log_y x = \frac{1}{\log_x y}$

$LHS = \frac{1}{\log_a (b+c)} + \frac{1}{\log_a (c-b)}$

$= \frac{\log_a (c-b) + \log_a (c+b)}{\log_a (c+b) \log_a (c-b)} = \frac{\log_a (c^2 - b^2)}{\log_a (c+b) \log_a (c-b)} = \frac{\log_a (a^2)}{\log_a (c+b) \log_a (c-b)}$

$= 2 \log_{c+b} a \cdot \log_{c-b} a$

Ex. 32 *If* $a^x = b, b^y = c, c^z = a, x = \log_b a^{k_1}, y = \log_c b^{k_2}, z = \log_a c^{k_3}$, *find the minimum value of* $\boxed{3k_1 + 6k_2 + 12k_3}$.

$\boxed{a^x = b \quad b^y = c \quad c^z = a}$

$x = \log_b a^{k_1} \quad y = \log_c b^{k_2}$

$z = \log_a c^{k_3}$

$x = \log_b a$ $y = \log_c b$

$z = \log_a c^{k_3}$

$(b^y)^z = a \Rightarrow b^{yz} = a$

$a^{xyz} = a$ $\qquad xyz = 1$ — (1)

$x = k_1 \log_b a$ $\quad y = k_2 \log_c b$

$z = k_3 \log_a c$

$xyz = k_1 k_2 k_3 \log_b a \log_c b \log_a c$

$1 = k_1 k_2 k_3$ — (2)

AM $\geq$ GM.

$$\frac{3k_1 + 6k_2 + 12k_3}{3} \geq (3k_1 6k_2 12k_3)^{1/3}$$

$$\frac{3k_1 + 6k_2 + 12k_3}{3} \geq (216 k_1 k_2 k_3)^{1/3}$$

$$\frac{3k_1 + 6k_2 + 12k_3}{3} \geq 6.$$

$3k_1 + 6k_2 + 12k_3 \geq 18$

• **Ex. 33** *If* $x = 1 + \log_a bc, y = 1 + \log_b ca, z = 1 + \log_c ab$, *prove that* $xyz = xy + yz + zx$.

$\downarrow$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

$$x = 1 + \log_a bc = \log_a a + \log_a bc = \log_a abc$$

$$\frac{1}{x} = \log_{abc} a$$

$$\frac{1}{y} = \log_{abc} b$$

$$\frac{1}{z} = \log_{abc} c$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \log_{abc} a + \log_{abc} b + \log_{abc} c = \log_{abc}(abc) = 1$$

• **Ex. 36** *Find the square of the sum of the roots of the equation* $\log_2 x \cdot \log_3 x \cdot \log_5 x = \log_2 x \cdot \log_3 x + \log_3 x \cdot \log_5 x + \log_5 x \cdot \log_2 x$.

$\log_2 x = a$ $\quad \log_3 x = b$ $\quad \log_5 x = c.$

$\frac{1}{a} = \log_x 2$ $\quad \frac{1}{b} = \log_x 3$ $\quad \frac{1}{c} = \log_x 5$

$\log_2 x = a \quad \log 3 \quad \log 5$

$\frac{1}{a} = \log_x 2 \quad \frac{1}{b} = \log_x 3 \quad \frac{1}{c} = \log_x 5.$

$abc = ab + bc + ca.$

$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$

$abc = abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

$\log_x 2 + \log_x 3 + \log_x 5 = 1$

$abc\left[\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1\right] = 0.$

$\log_x 30 = 1$

$30 = x$

$\boxed{a = 0, \; b = 0, \; c = 0.} \Rightarrow x = 1$

$30 + 1 = 31$

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• **Ex. 37** *Given that* $\log_2 a = \lambda, \log_4 b = \lambda^2$ *and* $\log_{c^2}(8) = \frac{2}{\lambda^3 + 1}$, *write* $\log_2\left(\frac{a^2 b^5}{c^4}\right)$ *as a function of* 'λ' $(a, b, c > 0, c \neq 1)$.

$\log_4 b = \lambda^2.$

$\frac{\log b}{2\log 2} = \lambda^2.$

$\log_2 a^2 + \log_2 b^5 - \log_2 c^4$

$\log_2 b = 2\lambda^2.$

$= 2\lambda + 5 \times 2\lambda^2 - 4 \times \left(\frac{3\lambda^3 + 3}{4}\right)$

$\log_8 c^2 = \frac{\lambda^3 + 1}{2}.$

$= 2\lambda + 10\lambda^2 - 3\lambda^3 - 3.$

$\frac{2\log c}{3\log 2} = \frac{\lambda^3 + 1}{2}.$

$\log_2 c = \frac{3\lambda^3 + 3}{4}.$

Ex. 40 *Solve the following equations for x and y*

$\log_{100}|x+y| = \frac{1}{2}, \log_{10} y - \log_{10}|x| = \log_{100} 4.$

$|x+y| = 100^{1/2} = 10.$

$\Rightarrow x+y = \pm 10$. — (1)

$x<0$

$x - 2x = \pm 10$

$-x = \pm 10$

$x = \mp 10.$

$x = -10$

$x>0$

$x + 2x = \pm 10.$

$x = \cancel{\pm}\frac{10}{3}.$

$x = \frac{10}{3}.$

$\log_{10}\left(\frac{y}{|x|}\right) = \frac{\cancel{2}\log 2}{\cancel{2}\log 10} = \log_{10} 2.$

$\frac{y}{|x|} = 2.$

$x<0$

$\frac{y}{-x} = 2.$

$\boxed{y = -2x}$

$y = 20.$

$x>0.$

$\frac{y}{x} = 2.$

$y = 2x.$

$y = \frac{20}{3}.$

$(-10, 20)$

$\left(\frac{10}{3}, \frac{20}{3}\right)$

Ex. 43 *Find all real numbers x which satisfy the equation* $2\log_2(\log_2 x) + \log_{1/2}\log_2(2\sqrt{2}x) = 1.$

$\log_2 x = a.$

$2\log_2 a + \log_{1/2}\left[\log_2 2\sqrt{2} + \log_2 x\right] = 1$

$2\log_2 a + \log_{\frac{1}{2}}\left[\frac{3}{2} + a\right] = 1$

$2\log_2 a + \frac{\log\left(\frac{3}{2}+a\right)}{\log\frac{1}{2}} = 1$

$2\log_2 a - \log_2\left(3/2 + a\right) = 1$

$\log_2\left(\frac{a^2}{\frac{3}{2}+a}\right) = 1$

$\frac{a^2}{\frac{3}{2}+a} = 2.$

$a^2 = 2a + 3.$

$a^2 - 2a - 3 = 0.$

$(a-3)(a+1) = 0.$

$a = 3, -1.$

$\log_2 x = 3, -1$

$2\sqrt{2} = 2^{3/2}.$

$\log_2 2\sqrt{2} = \log_2 2^{3/2} = \frac{3}{2}.$

$\log\frac{1}{2} = \log 1 - \log 2 = -\log 2.$

$(a-3)(a+1)=0$

$a=3,-1$

$\log_2 x = 3, -1$

$\boxed{x = 8, \frac{1}{2}}$

• Ex. 45 *Prove that*

$$2^{\left(\sqrt{\log_a \sqrt[4]{ab} + \log_b \sqrt[4]{ab}} - \sqrt{\log_a \sqrt[4]{b/a} + \log_b \sqrt[4]{a/b}}\right)\sqrt{\log_a b}} = \begin{cases} 2, & \boxed{b \ge a > 1} \\ 2^{\log_a b}, & 1 < b < a \end{cases}$$

$\sqrt{(-5)^2} = 5$

$\log_a (ab)^{\frac{1}{4}} + \log_b (ab)^{\frac{1}{4}}$

$= \log (ab)^{1/4} \left[\frac{1}{\log a} + \frac{1}{\log b}\right]$

$= \frac{\log (ab)^{1/4} \log(ab)}{\log a \cdot \log b}$

$= \frac{1}{4} \frac{[\log ab]^2}{\log a \log b}$

$= \frac{1}{4} \frac{\log(ab)}{\log a} \frac{\log(ab)}{\log b}$

$= \frac{1}{4}[1+\log_a b][1+\log_b a]$

$= \frac{1}{4}\left[1+\frac{1}{\log_b a}\right][1+\log_b a]$

$= \frac{1}{4} \cdot \frac{1}{\log_b a}[\log_b a + 1]^2$ — (1)

$\log_a \left(\frac{b}{a}\right)^{\frac{1}{4}} + \log_b \left(\frac{a}{b}\right)^{\frac{1}{4}}$

$= \frac{1}{4}[\log_a b - 1] + \frac{1}{4}[\log_b a - 1] = \frac{1}{4}[\log_a b + \log_b a - 2] = \frac{1}{4}\left[\log_b a + \frac{1}{\log_b a} - 2\right]$

$= \frac{1}{4} \cdot \frac{1}{\log_b a}[\log_b a - 1]^2$ — (2)

$\sqrt{(1)} - \sqrt{(2)}$

$= \frac{1}{2} \cdot \frac{1}{(\log_b a)^{1/2}}[\log_b a + 1] - \frac{1}{2(\log_b a)^{1/2}}|\log_b a - 1|$

$2^{\frac{1}{2}\log_a b [\log_b a + 1 - |\log_b a - 1|]}$

$\log_b a < 1 \rightarrow |\log_b a - 1| = -(\log_b a - 1)$

$= -\log_b a + 1$

$\log_b a + 1 - (-\log_b a + 1)$

$= 2\log_b a$

90. The value of

$$6 + \log_{3/2}\left(\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\cdots}}}\right) \text{ is}$$

[IIT-JEE 2012, 4M]

89. Let (x_0, y_0) be solution of the following equations $(2x)^{\ln 2} = (3y)^{\ln 3}$ and $3^{\ln x} = 2^{\ln y}$, then x_0 is

[IIT-JEE 2011, 3M]

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 6

[IIT-JEE 2012, 4M]

91. If $3^x = 4^{x-1}$, then x equals **[JEE Advanced 2013, 3M]**

(a) $\dfrac{2\log_3 2}{2\log_3 2 - 1}$ (b) $\dfrac{2}{2 - \log_2 3}$

(c) $\dfrac{1}{1 - \log_4 3}$ (d) $\dfrac{2\log_2 3}{2\log_2 3 - 1}$