How do we estimate σ^2: [from sample]:

From unknown popln $Var(u_i) = \sigma^2$. [by assumption]

A reasonable guess to estimate σ^2 would be $Var(e) = \frac{1}{n}\sum(e_i - \bar{e})^2$

$$= \frac{1}{n}\sum e_i^2 \; (\because \bar{e} = 0)$$

To confirm we need check: $E(\sum e_i^2)$ [to confirm unbiaseness]

Result: $E(\sum e_i^2) = (n-2)\sigma^2$.

$$\Rightarrow E\left(\underbrace{\frac{\sum e_i^2}{(n-2)}}_{\hat{\sigma}^2}\right) = \sigma^2. \Rightarrow E(\hat{\sigma}^2) = \sigma^2, \text{ where } \hat{\sigma}^2 = \frac{\sum e_i^2}{(n-2)}$$

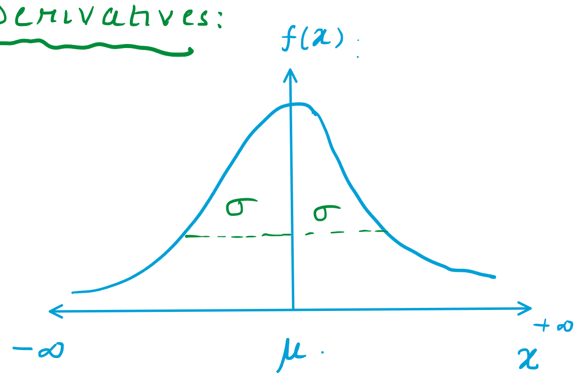
$$\therefore \widehat{Var}(\hat{\beta}) = \frac{\hat{\sigma}^2}{\sum x_i^2}, \quad \widehat{Var}(\hat{\alpha}) = \frac{\hat{\sigma}^2 \sum X_i^2}{n\sum x_i^2}, \text{ where } \hat{\sigma}^2 = \frac{\sum e_i^2}{(n-2)}.$$

Digression: Normal Distributions & its Derivatives:

$\therefore$ r.v $X \sim N(\mu, \sigma^2)$.

$\Rightarrow E(X) = \mu$, $Var(X) = \sigma^2$.

$\Rightarrow$ Symmetric about μ.



(i) Standard Normal Distribution:

$X \sim N(\mu, \sigma^2)$.

construction of the new r.v.

$\therefore$ Define: $\tau = \left(\frac{X-\mu}{\sigma}\right) \sim N(0,1)$.

Find: $P[X \leq c] = \int_{-\infty}^{c} f(x)\,dx$ [By computation]

Find: $P[X \leq c] = \int_{-\infty}^{c} f(x)\,dx$ [By computation]

(OR) $\rightarrow P\left[\left(\frac{X-\mu}{\sigma}\right) \leq \frac{c-\mu}{\sigma}\right] = P\left[\tau \leq \underbrace{\left(\frac{c-\mu}{\sigma}\right)}_{=k}\right] = \int_{-\infty}^{\left(\frac{c-\mu}{\sigma}\right)} f(\tau)\,d\tau$.

$= P[\tau \leq k]$

$= \Phi(k)$ [use table to find this value] $\longrightarrow$

Standard Normal Table

τ	$P[\tau \leq k]$
$-\infty$	⋮
⋮	⋮
-1	⋮
0	⋮
1	⋮
⋮	⋮
$+\infty$	

Note: Suppose we have a popln $X \sim N(\mu, \sigma^2)$

r.s: $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$\Rightarrow$ Here every $X_i \sim N(\mu, \sigma^2)\ \forall i$

$\Rightarrow$ Consider $X_1 = 1^{st}$ sampling unit.

$\therefore$ Any popln unit can qualify as the 1^{st} sampling unit

$\Rightarrow X_1 \sim N(\mu, \sigma^2)$

By same logic: $X_2 \sim N(\mu, \sigma^2) \ldots\ldots X_n \sim N(\mu, \sigma^2)$.

$\therefore X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$.

$\Rightarrow X_i \sim N(\mu, \sigma^2)\ \forall i$

$\Rightarrow$ Define $\tau_i = \frac{X_i - \mu}{\sigma} \sim N(0,1)\ \forall i$.

(ii) Chi-square Distribution:

$X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$.

Define $\tau_i = \left(\frac{X_i - \mu}{\sigma}\right) \sim N(0,1)$.

Define: $Y = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$ [chi-sq distribution with n degrees of freedom].

$\rightarrow$ constructed r.v