Expectation and Variance of a Random Variable :

Expectation :

$$E(x) = \begin{cases} \sum\limits_{x} x \, \underbrace{P(X=x)}_{\text{p.m.f} \,\to\, \text{probability mass funxn}} , & \text{if } X \text{ is discrete} \\ \int\limits_{x} x \, \underbrace{f(x)}_{\text{p.d.f} \,\to\, \text{probability density funxn}} \, dx , & \text{if } X \text{ is continuous} \end{cases}$$

Properties :

1) If $X = c$ (const), then $E(X) = c$.
2) If $Y = cX$ (c = const), then $E(Y) = c\,E(X)$
3) If $Y = a + bX$ where $a, (b \neq 0)$ are consts, then $E(Y) = a + b\,E(X)$ [Combining ① & ②]
4) $Z = X + Y$, then $E(Z) = E(X) + E(Y)$.
   → Sum Law of Expectation.
5)* $Z = X \cdot Y$, then $E(Z) = E(X) \cdot E(Y)$, if X and Y are independent random variables.
   → Product Law of Expectation.

Variance : $V(X) = E(X^2) - \{E(X)\}^2 = E(X^2) - E^2(X) \;\to\; \frac{1}{n}\sum x_i^2 - \bar{x}^2$

Alternative : $V(X) = E[X - E(X)]^2 \;\to\; \frac{1}{n}\sum (x_i - \bar{x})^2$

In general,

$$E[g(x)] = \begin{cases} \sum\limits_{x} g(x)\, P(X=x) , & \text{if } X \text{ is discrete RV} \\ \int\limits_{x} g(x)\, f(x)\, dx , & \text{if } X \text{ is continuous RV} \end{cases}$$

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$$E(x^2) = \begin{cases} \sum_x x^2 \, P(X=x), & \text{if } X \text{ is discrete RV} \\ \int_x x^2 f(x)\,dx, & \text{if } X \text{ is continuous RV} \end{cases}$$

Consider, $g(x) = [X - E(x)]^2$

$$E[g(x)] = \underbrace{E[X - E(X)]^2}_{\text{Alternative } V(X) \text{ formula}} = \begin{cases} \sum_x \{x - E(x)\}^2 P(X=x), & \text{if } X \text{ is discrete RV} \\ \int_x \{x - E(x)\}^2 f(x)\,dx, & \text{if } X \text{ is continuous RV} \end{cases}$$

Properties:

1) If $X = c$ (const), then $V(X) = 0$ w.p 1.
2) If $Y = bX$, (b = const), then $V(Y) = b^2 V(X)$

Note that, If $Y = a + bX$, where, $a, (b \neq 0)$ are consts, then $V(Y) = b^2 V(X)$, as variance does not depends on change of origin.

$V(Y) = V(a + bX)$

$Y = a + bX$

$Y^2 = (a + bX)^2$

$= a^2 + b^2X^2 + 2abX$.

$\underline{\underline{V(Y)}} = E(Y^2) - E^2(Y)$

$= E(a^2 + b^2X^2 + 2abX) - \{E(a + bX)\}^2$

$= a^2 + b^2 E(X^2) + 2ab\,E(X) - \{a + b\,E(X)\}^2$

$= \cancel{a^2} + b^2 E(X^2) + 2ab\,E(X) - \{\cancel{a^2} + b^2 E^2(X) + 2ab\,E(X)\}$

$$- \{\cancel{a^2} + b^2 E^2(x) + 2ab\ E(x)\}$$

$$= b^2 \{E(x^2) - E^2(x)\} = \underline{\underline{b^2\ V(x)}}$$

Covariance:

$$\mathrm{Cov}(x, y) = E(xy) - E(x) \cdot E(y)$$

Note that, If x and y are two independent R.V's, then $E(xy) = E(x) \cdot E(y)$, and hence $\mathrm{cov}(x, y) = 0$. Converse is not true.

1) If $E(x) = 4$ and $V(x) = 9$, then find $E(x^2)$.

Ans: $V(x) = 9 \Rightarrow E(x^2) - E^2(x) = 9$

$$\Rightarrow E(x^2) - 16 = 9$$

$$\Rightarrow E(x^2) = 9 + 16 = 25$$

2) If a Rv. X assumes two values (-2) and 1 such that $2 \cdot P(X = -2) = \underbrace{P(X = 1)}_{K}$. then find the value of $E(x)$.

X	$P(X=x)$
-2	$P(X=-2)$
1	$P(X=1)$.

$$E(x) = \sum_x x\, P(X = x)$$

$$= (-2) \cdot P(X = -2) + 1 \cdot P(X = 1)$$

$$= (-2)\, P(X = -2) + 1 \cdot 2 \cdot P(X = -2)$$

$$= P(X = -2) \times 0 = 0.$$

3) Consider two independent R.V's X and Y. with $V(x) = 6$ and $V(Y) = 10$. Find $V(X - Y)$.

Ans: $V(X \pm Y) = V(X) + V(Y) \pm 2\mathrm{Cov}(x, Y)$.

$$V(aX \pm bY) = a^2 V(x) + b^2 V(Y) \pm 2ab\,\mathrm{Cov}(x, y)$$

$$V(X - Y) = V(x) + V(Y) - 2\underbrace{\mathrm{Cov}(x, y)}$$

$$V(x-y) = V(x) + V(y) - 2\underbrace{Cov(x,y)}_{=0} \quad \text{as } x \,\&\, y \text{ are independent}$$

$$= 10 + 6 = 16.$$

4) For two Random Variables x and y, the following informations are given.

$E(x) = 8$, $E(y) = 6$.

$V(x) = 16$, $V(y) = 36$, $\rho_{xy} = 0.5$

Find, — i) $E(xy)$; iv) $V(5x+2y)$

ii) $Cov(x, x+y)$; v) $Cov(x, x-y)$

iii) $V(2x-5y)$; vi) $Corr(2x+3y, 2x-3y)$

Ans.

	X	Y
Expectation	8	6
Variance	16	36

$\rho_{xy} = 0.5$

i) $$\rho_{xy} = \frac{Cov(x,y)}{\sqrt{V(x) \cdot V(y)}}$$

$$\Rightarrow 0.5 = \frac{E(xy) - E(x) \cdot E(y)}{\sqrt{V(x) \cdot V(y)}}$$

$$\Rightarrow 0.5 = \frac{E(xy) - 8 \times 6}{4 \times 6}$$

$$\Rightarrow \frac{1}{2} \times 4 \times 6 = E(xy) - 8 \times 6$$

$$\Rightarrow 12 = E(xy) - 48$$

$$\Rightarrow E(xy) = 48 + 12 = 60$$

ii) $Cov(x, x+y)$

$$= V(x) + Cov(x,y)$$

$$= 16 + \{E(xy) - E(x) \cdot E(y)\}$$

$$= 16 + \{60 - 8 \times 6\}$$

$$= 16 + 12$$

v) $Cov(x, x-y)$

$$= V(x) - Cov(x,y)$$

$$= 16 - \{E(xy) - E(x) \cdot E(y)\}$$

$$= 16 - \{60 - 8 \times 6\}$$

$$= 16 - 12$$

$= 28$

iii) $V(2X - 5Y)$

$= 2^2 V(X) + 5^2 V(Y) - 2\times 2\times 5\, Cov(X,Y)$

$= (4\times 16) + (25\times 36) - (20\times 12)$

$=$

$= 4$

iv) $V(5X + 2Y)$

$= 5^2 V(X) + 2^2 V(Y)$
$\quad + 2\times 5\times 2\, Cov(X,Y)$

$= (25\times 16) + (4\times 36)$
$\quad + (20\times 12)$

vi) $Corr(2X+3Y,\ 2X-3Y)$

$$= \frac{Cov(2X+3Y,\ 2X-3Y)}{\sqrt{V(2X+3Y)\,V(2X-3Y)}} =$$

$$Cov(2X+3Y,\ 2X-3Y) = 4V(X) - 6\cancel{Cov(X,Y)} + 6\cancel{Cov(X,Y)} - 9V(Y)$$