Cobb_Douglas utility function.

Q. $U(x_1, x_2) = x_1^a x_2^b$. Find the optimal soln (Marshallian demand curves for x_1 & x_2).

Slope of IC = slope of BL (at optimal) $\Rightarrow \boxed{\frac{MU_1}{MU_2} = \frac{P_1}{P_2}}$ → optimization condition

$\frac{MU_1}{MU_2}$ → MRS.

$$MRS = \frac{MU_1}{MU_2} = \frac{\partial U/\partial x_1}{\partial U/\partial x_2} = \frac{a x_1^{a-1} x_2^b}{b x_1^a x_2^{b-1}} = \left(\frac{a}{b}\right)\left(\frac{x_2}{x_1}\right)$$

Now, opt: $MRS = \frac{P_1}{P_2} \Rightarrow \left(\frac{a}{b}\right)\left(\frac{x_2}{x_1}\right) = \frac{P_1}{P_2} \Rightarrow \boxed{x_2 = \left(\frac{b}{a}\right)\left(\frac{P_1}{P_2}\right) \cdot x_1}$

∴ Now. Budget constraint: $M = P_1 x_1 + P_2 x_2$

$$M = P_1 x_1 + \cancel{P_2}\left(\frac{b}{a}\right)\left(\frac{P_1}{\cancel{P_2}}\right) \cdot x_1$$

$$M = P_1 x_1 + \frac{b}{a} \cdot P_1 x_1.$$

$$M = \left(1 + \frac{b}{a}\right) \cdot P_1 x_1.$$

$$M = \left(\frac{a+b}{a}\right) P_1 x_1$$

$$x_1^* = \left(\frac{a}{a+b}\right)\left(\frac{M}{P_1}\right)$$ ... Marshallian demand for Good 1.

Now $x_2^* = \left(\frac{b}{a}\right)\left(\frac{P_1}{P_2}\right) x_1^* = \left(\frac{b}{a+b}\right)\left(\frac{M}{P_2}\right)$ ... Marshallian demand for Good 2

Eg: $u(x_1, x_2, x_3) = x_1^a x_2^b x_3^c$

$$x_1^* = \left(\frac{a}{a+b}\right)\left(\frac{M}{P_1}\right) \qquad x_2^* = \left(\frac{b}{a+b}\right)\left(\frac{M}{P_2}\right)$$

$$\frac{P_1 x_1^*}{M} = \left(\frac{a}{a+b}\right) \qquad \frac{P_2 x_2^*}{M} = \left(\frac{b}{a+b}\right) \qquad U = x_1^a x_2^b.$$

Total exp on Good 1.

Share of exp on Good 1

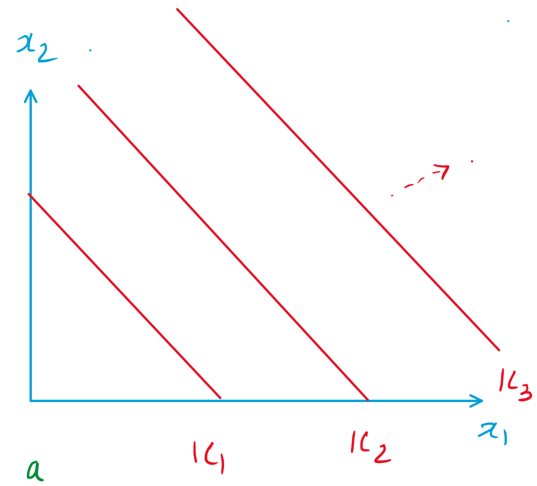
Share of exp on Good 2

→ Perfect substitutes.

> Share of exp on Good 1

Perfect substitutes.

B. Consider $u = ax_1 + bx_2$. Find the marshallian demands for Good 1 & Good 2.



$$u = ax_1 + bx_2$$

Diff: $du = a\,dx_1 + b\,dx_2$.

For IC: $du = 0 \Rightarrow 0 = a\,dx_1 + b\,dx_2$

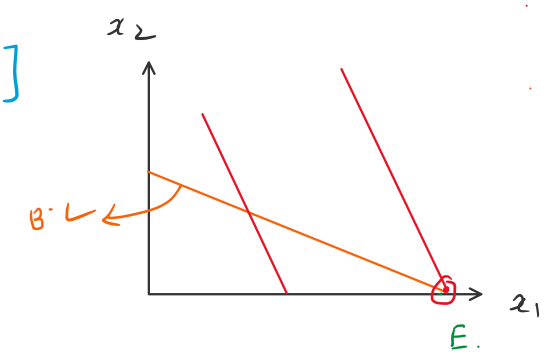
$$b\,dx_2 = -a\,dx_1$$

$$\left.\frac{dx_2}{dx_1}\right|_{IC} = -\frac{a}{b} \;\rightarrow\; \left|\frac{dx_2}{dx_1}\right|_{IC} = \frac{a}{b}$$

Budget Line: $M = P_1x_1 + P_2x_2$.

$$\left.\frac{dx_2}{dx_1}\right|_{BL} = -\frac{P_1}{P_2} \rightarrow \left|\frac{dx_2}{dx_1}\right|_{BL} = \frac{P_1}{P_2}$$

Case I: $\frac{P_1}{P_2} < \frac{a}{b}$. [BL is flatter than IC]



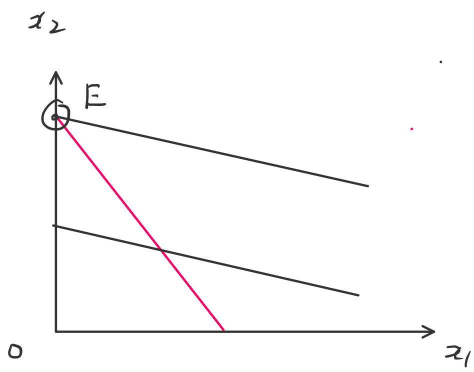
$$x_2^* = 0$$

B·L $\Rightarrow$ $M = P_1x_1 + P_2x_2$.

$$M = P_1x_1 \Rightarrow x_1^* = \left(\frac{M}{P_1}\right)$$

$$(x_1^*, x_2^*) = \left(\frac{M}{P_1}, 0\right)$$

Case II: $\frac{P_1}{P_2} > \frac{a}{b}$. [B·L is steeper than IC]

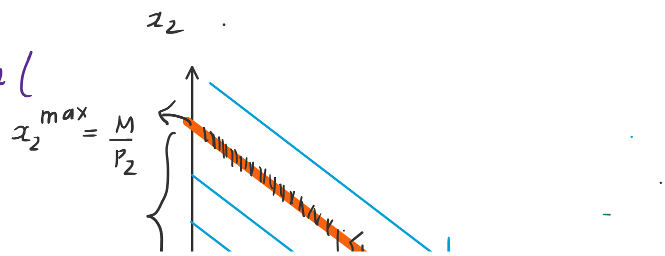


$$x_1^* = 0, \quad x_2^* = \frac{M}{P_2}.$$

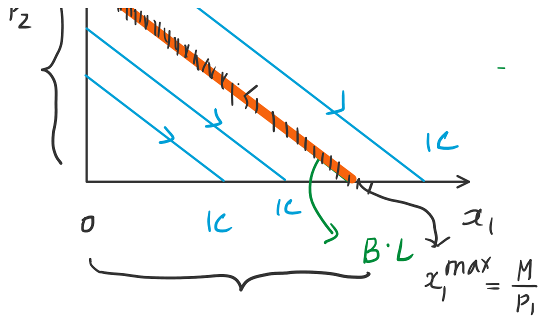
$$(x_1^*, x_2^*) = \left(0, \frac{M}{P_2}\right)$$

Case III: $\frac{P_1}{P_2} = \frac{a}{b}$ [BL, IC have same slope]

Any pt on the B·L can be a solution (optimal solution)



$(x_1^*, x_2^*) \in BL$ [optimal solution].



$\therefore$ Optimal solution:

$$\left\{ (x_1^*, x_2^*) \;\middle|\; x_1^* \in \left[0, \frac{M}{P_1}\right], \; x_2^* \in \left[0, \frac{M}{P_2}\right], \; M = P_1 x_1^* + P_2 x_2^* \right\}$$

Marshallian demand curves:

$$x_1^* = \begin{cases} \frac{M}{P_1}, & \frac{P_1}{P_2} < \frac{a}{b}. \\ 0, & \frac{P_1}{P_2} > \frac{a}{b} \\ \in \left[0, \frac{M}{P_1}\right], & \frac{P_1}{P_2} = \frac{a}{b} \end{cases} \qquad x_2^* = \begin{cases} 0, & \frac{P_1}{P_2} < \frac{a}{b}. \\ \frac{M}{P_2}, & \frac{P_1}{P_2} > \frac{a}{b}. \\ \in \left[0, \frac{M}{P_2}\right], & \frac{P_1}{P_2} = \frac{a}{b}. \end{cases}$$

$\Rightarrow \frac{P_1}{P_2} < \frac{a}{b}$: $x_1^* = \frac{M}{P_1}$, $x_2^* = 0$

$\Rightarrow \frac{P_1}{a} < \frac{P_2}{b} \;\Rightarrow\; \frac{a}{P_1} > \frac{b}{P_2} \;\Rightarrow\; \frac{MU_1}{P_1} > \frac{MU_2}{P_2}.$

$\frac{P_1}{MU_1} < \frac{P_2}{MU_2}$.

additional benefit from Good 1

additional cost for Good 1

$\left. \begin{array}{l} u = a x_1 + b x_2 \\ M = P_1 x_1 + P_2 x_2 \end{array} \right\}$ ✓

$MU_1 = \frac{\partial u}{\partial x_1} = a$

$MU_2 = \frac{\partial u}{\partial x_2} = b$

$\Rightarrow$ Buy the good that is relatively cheaper

$\frac{MU_1}{P_1}$ = net benefit obtained from Good 1

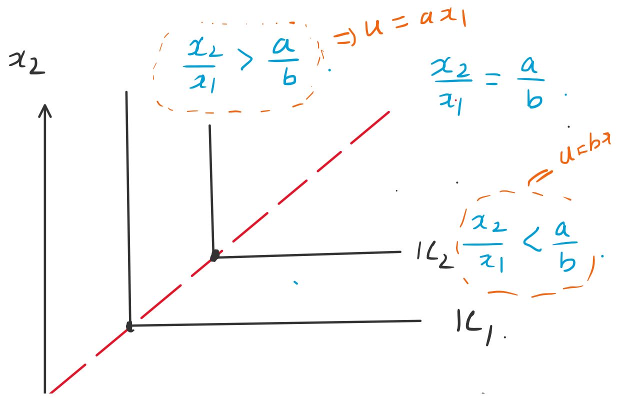
Q. $u = \min\{a x_1, b x_2\}$, $a, b > 0$. Find the marshallian demands for Good 1, Good 2. $\frac{x_2}{x_1} > \frac{a}{b}$.

If $a x_1 < b x_2 \Rightarrow \bar{u} = a x_1$.

If $b x_2 < a x_1 \Rightarrow \bar{u} = b x_2$.

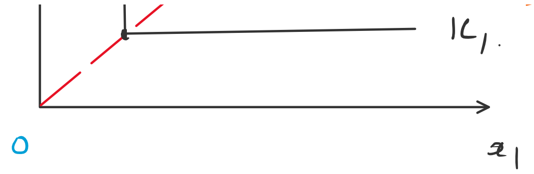
If $b x_2 = a x_1 \Rightarrow \bar{u} = b x_2 = a x_1$.

$\left(\frac{x_2}{x_1} = \frac{a}{b}\right)$



If $\boxed{b x_2 = a x_1}$ => ... $b x_2 = a x_1$

$$\left(\frac{x_2}{x_1} = \frac{a}{b}.\right)$$



$$\left.\frac{dx_2}{dx_1}\right|_{IC} = $$
$= 0$

If: $\frac{x_2}{x_1} > \frac{a}{b}$: $u = a x_1 \Rightarrow du = a\, dx_1$

$$\Rightarrow 0 = a\, dx_1$$

$$\Rightarrow dx_1 = 0$$

$$\frac{dx_2}{dx_1} \to \infty.$$

If $\frac{x_2}{x_1} < \frac{a}{b}$: $u = b x_2 \Rightarrow du = b\, dx_2$

$$\Rightarrow 0 = b\, dx_2$$

$$\Rightarrow dx_2 = 0$$

$$\frac{dx_2}{dx_1} = 0$$