$k > n \Rightarrow$ set will always be l.d

Note: (i) Basis of a vector space need not be unique.

(ii) Consider $V \subseteq \mathbb{R}^n$. Consider a set $B = \{ \underset{\sim}{b_1}, \underset{\sim}{b_2}, \cdots, \underset{\sim}{b_k} \}$.

[Check if B is a valid basis of the vector space].

For qualifying to be a valid basis $k \leq n$

Eg: $V = \mathbb{R}^3$: [For valid basis of $\mathbb{R}^3$, No. of vectors in basis ≤ 3].

$$B = \left\{ \underset{3\times1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{3\times1}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{3\times1}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\} \Rightarrow \text{No. of vectors} = 3 \; (=k),$$

$$B' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \underset{3\times1}{\begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}} \right\} \Rightarrow \text{No. of vectors} = 4 \; (=k) >$$

Dimension of vectors $= 3 \; (=n)$

As for B', $k > n \Rightarrow$ set of vectors B' is linearly dependent, hence cannot form a basis.

$$B' = \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\underset{\sim}{b_1}}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\underset{\sim}{b_2}}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\underset{\sim}{b_3}}, \underbrace{\begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}}_{\underset{\sim}{b_4}} \right\}.$$

$\therefore$ $c_1 \underset{\sim}{b_1} + c_2 \underset{\sim}{b_2} + c_3 \underset{\sim}{b_3} + c_4 \underset{\sim}{b_4} = 0 \Rightarrow$ solve c_1, c_2, c_3, c_4 :-

$$\begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Q. Find the basis & dim of $S = \{ (x, y, z) \in \mathbb{R}^3 : \underbrace{x + y + z = 0}_{\text{Restriction}} \}$.

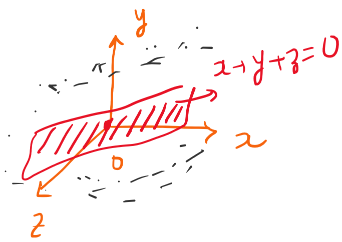
$\Rightarrow$ Max^m possible dimension of $S = 3$.

(*) dim = Max possible dim − No. of restrictions $= 3 - 1 = 2$

Consider $(x, y, z) \in S \Rightarrow x + y + z = 0$.

$\Rightarrow z = -x - y$

$$(x, y, z) = (x, y, -x-y)$$
$$= x(1, 0, -1) + y(0, 1, -1)$$



any $(x, y, z) \in S$ can be expressed as a linear combination

$(1,0,-1),\ (0,1,-1)$ $\quad B=\{(1,0,-1),\ (0,1,-1)\}$, with $\underset{\sim}{b_1} = (1,0,-1)$ and $\underset{\sim}{b_2} = (0,1,-1)$.

$\therefore$ For l.i. $\quad c_1(1,0,-1)+c_2(0,1,-1)=(0,0,0)$

$$(c_1,\ c_2,\ -c_1-c_2)=(0,0,0)$$

$$c_1=c_2=0 \Rightarrow \underset{\sim}{b_1},\ \underset{\sim}{b_2} \text{ are l.i}$$

$B=\{(1,0,-1),\ (0,1,-1)\}$ forms a basis.

$\dim(S)=2$.

Q. Consider subspace $W=\{(x,y,z)\in\mathbb{R}^3 \mid 2x-y+3z=0,\ x+y+z=0\}$
Find the basis & dim of W.

No. of restriction $=2$ $\quad(*)$
Max possible dim $=3$
True dim $=3-2=1$

Consider $(x,y,z)\in W$.

$$\therefore \left.\begin{aligned} 2x-y+3z&=0 \\ x+y+z&=0 \end{aligned}\right\} \cdots\ x,y,z$$

Method of cross multiplication:

$$\frac{x}{+\begin{vmatrix}-1 & 3\\ 1 & 1\end{vmatrix}}=\frac{y}{-\begin{vmatrix}2 & 3\\ 1 & 1\end{vmatrix}}=\frac{z}{+\begin{vmatrix}2 & -1\\ 1 & 1\end{vmatrix}}=k.$$

$$\frac{x}{-4}=\frac{y}{1}=\frac{z}{3}=k \Rightarrow \left.\begin{aligned} x&=-4k \\ y&=k \\ z&=3k \end{aligned}\right\}$$

$$(x,y,z)\in W \Rightarrow (-4k,\ k,\ 3k)$$

$$\Rightarrow k(-4,\ 1,\ 3).$$

$B=\{(-4,1,3)\}$ $\Rightarrow$ Valid basis.
l.i $\qquad \dim(W)=1$.

HW · Check if $B = \{(2,-1,0), (3,5,1), (1,1,2)\}$ forms a valid basis of $V = \mathbb{R}^3$.