Leibnitz Rule : [Finding the derivative of an integral] .

Suppose we have a fn $f(x)$ continuous over $[a, b]$ and fns $u(x)$ and $v(x)$ differentiable over $[a, b]$, then .

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right] = f\{v(x)\}\cdot v'(x) - f\{u(x)\}\cdot u'(x)$$

Eg: $y = f(x) = \int_{x^2}^{x^3} \frac{1}{\ln t}\cdot dt$; $x > 0$. Find $\frac{dy}{dx}$

$$\frac{d}{dx}\left[\int_{x^2}^{x^3} \underbrace{\frac{1}{\ln t}}_{f(t)} dt\right] = \frac{1}{\ln(x^3)}\cdot\frac{d}{dx}(x^3) - \frac{1}{\ln(x^2)}\cdot\frac{d}{dx}(x^2)$$

$$= \frac{x^2 - x}{\ln x}$$

Q. Let $f : (0, \infty) \to (0, \infty)$ be a differentiable fn satisfying $x\int_0^x (1-t) f(t)\,dt = \int_0^x t\cdot f(t)\,dt \quad \forall x \in \mathbb{R}^+$ and $f(1) = 1$. Find $f(x)$

Given: $x\int_0^x (1-t) f(t)\,dt = \int_0^x t\, f(t)\,dt$.

Diff : $\int_0^x (1-t) f(t)\,dt + x\cdot\left[(1-x) f(x)\cdot 1 - 0\right] = x f(x)\cdot 1 - 0$.

$$\int_0^x (1-t) f(t)\,dt + \underbrace{x(1-x) f(x)} = x f(x) .$$

$$\int_0^x (1-t) f(t)\,dt = x f(x) - x(1-x) f(x)$$

$x - x(1-x)$

$\cancel{x} - \cancel{x} + x^2$

$$\int_0^x (1-t) f(t)\,dt = x^2 f(x)$$

Diff: $(1-x)\cdot f(x) = x^2 f'(x) + f(x)(2x)$.

$(1-3x)\, f(x) = x^2 f'(x)$

Diff.

$$(1-3x)\, f(x) = x^2 f'(x)$$

$$\frac{f'(x)}{f(x)} = \frac{1-3x}{x^2}$$

Int:
$$\int \frac{f'(x)}{f(x)}\, dx = \int \frac{1-3x}{x^2}\, dx$$

$$\ln f(x) = -\frac{1}{x} - 3\ln x + c.$$

$f(1)=1 \Rightarrow$
$$\ln \underbrace{f(1)}_{=1} = -1 - 3\ln 1 + c$$

$$0 = -1 + c \Rightarrow c = 1$$

$$\Rightarrow \ln f(x) = -\frac{1}{x} - 3\ln x + 1$$

$$\Rightarrow f(x) = e^{-1/x - 3\ln x + 1} = \left(\frac{e^{1-1/x}}{x^3}\right)$$

Q. If $f(x) = \displaystyle\int_{x^2}^{x^2+1} e^{-t^2}\, dt$. Find the interval in which $f(x)$ is increasing.

$$f'(x) = e^{-(x^2+1)^2}(2x) - e^{-x^4}(2x)$$
$$= e^{-(x^4+2x^2+1)}(2x) - e^{-x^4}(2x)$$
$$= 2x\, e^{-x^4}\left(e^{-(2x^2+1)} - 1\right)$$

For increasing fn: $f'(x) \geq 0 \Rightarrow \underbrace{x}_{<0 \,/\, \geq 0}\ \underbrace{e^{-x^4}}_{>0} \underbrace{\left[e^{-(2x^2+1)} - 1\right]}_{<0} \geq 0.$

$$2x^2 \geq 0 \Rightarrow (2x^2+1) \geq 1.$$

$$-(2x^2+1) \leq -1$$

$$e^{-(2x^2+1)} \leq e^{-1}$$

$a > b$

$e^a > e^b.$

$$e^{-(2x^2+1)} \leq (e^{-1}-1) = \left(\frac{1}{e}-1\right) < 0.$$

$\therefore f'(x) \geq 0$ when $x < 0 \Rightarrow x \in (-\infty, 0]$.

Q. Find the points of extremum for $\displaystyle\int_0^{x^2} \frac{t^2-5t+4}{2+e^t}\,dt$.

Let $f(x) = \displaystyle\int_0^{x^2} \frac{t^2-5t+4}{2+e^t}\,dt$

$$f'(x) = 0 \quad \Rightarrow \quad \frac{2x(x^4-5x^2+4)}{2+e^{x^2}} = 0$$

$$x(x^4-5x^2+4) = 0$$

$x = 0$ or $x^4-5x^2+4 = 0$.

$$(x^2-1)(x^2-4) = 0$$

$$x = \pm 1,\ x = \pm 2.$$

$$x = 0, \pm 1, \pm 2$$

Q. Evaluate: $\displaystyle\lim_{x\to\infty} \frac{\left(\int_0^x e^{x^2}dx\right)^2}{\int_0^x e^{2x^2}dx}$ $\left[\frac{\infty}{\infty}\right]$ Indeterminate Form

as $x \to \infty$: $\displaystyle\int_0^\infty e^{x^2}dx$; $x^2 \geq 0$, $e^{x^2} \to \infty$ as $x \to \infty$.

Using L·Hopital's:-

$$\lim_{x\to\infty} \frac{2\left(\int_0^x e^{x^2}dx\right)\cdot\left[e^{x^2}\cdot(1) - e^0\cdot 0\right]}{e^{2x^2}\cdot 1}.$$

$$\lim_{x\to\infty} \frac{2\left(\int_0^x e^{x^2}dx\right)\cdot\left(e^{x^2}\right)}{\left(e^{2x^2}\right)}.$$

$$2 \lim \int^{x} e^{x^2}dx \qquad \left[\infty\right]$$

$$2 \quad \lim_{x\to\infty} \frac{\int_0^x e^{x^2}\,dx}{e^{x^2}} \qquad \left[\frac{\infty}{\infty}\right].$$

Using L·Hopital's Rule:

$$2 \cdot \lim_{x\to\infty} \cdot \frac{e^{x^2}}{e^{x^2}\cdot 2x} = \lim_{x\to\infty} \cdot \frac{1}{x} = 0.$$