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Number Theory

Let $a \geq b \geq c > 0$ be real numbers such that for all $n \in \mathbb{N}$, there exist triangles of side lengths a^n, b^n, c^n. Prove that the triangles are isosceles.

$a, b, c \to$ real numbers $\quad n \to$ Natural No. +ve integer.

$a \geq b \geq c \to a$ is the largest side.

Sum of the 2 smaller sides of a Δ > Largest side.

$$\frac{a^n}{a^n} < \frac{b^n + c^n}{a^n}$$

$$1 < \left(\frac{b}{a}\right)^n + \left(\frac{c}{a}\right)^n$$

$\boxed{b \leq a \qquad c \leq a}$

$b < a, c < a \qquad b = a \quad c = a$

when $n \to \infty$ $\quad \left(\frac{b}{a}\right)^n \to 0$ $\quad \left(\frac{c}{a}\right)^n \to 0$

+ve fraction $\begin{cases} 0 < \frac{b}{a} < 1 \\ 0 < \frac{c}{a} < 1 \end{cases}$

$1 < 0$ ✗

$1 < 1 + 1$

$1 < 2$

$b = c = a$

Let $a, b, c \in \mathbb{N}$ be such that

$$a^2 + b^2 = c^2 \text{ and } c - b = 1.$$

$\to \boxed{c = b + 1} \qquad c - 1 = b$

Prove that

$a^2 = c^2 - b^2 = (c-b)(c+b) = c + b = b + 1 + b = 2b + 1$

(i) a is odd, ✓

(ii) b is divisible by 4, ✓

(iii) $a^b + b^a$ is divisible by c.

$k \in \mathbb{N}$

$a = 2k + 1$

$a^2 = 4k^2 + 4k + 1 \leftarrow \boxed{a^2 = 2b + 1}$

$2b \to$ even. $\quad 2b + 1 \to$ odd.

a^2 is odd $\therefore a \to$ odd

$4k^2 + 4k + \cancel{1} = 2b + \cancel{1}$

$b = 2k^2 + 2k$

$\boxed{b = 2k(k+1)}$

b is divisible

$b = c - 1$

$b^2 = c^2 - a^2$

$(c-1)^2 = c^2 - a^2$

$c \to$ even, odd.

$c^2 - 2c + 1 = c^2 - a^2$

$\boxed{a^2 = 2c - 1}$

$a^b + b^a = a^{c-1} + (c-1)^a = a^{c-1} + c^a - a \cdot c^{a-1} + \frac{a(a-1)}{2} c^{a-2} - \dots - 1$

$$= (a^{c-1} - 1) + \underbrace{\left[c^a - a \cdot c^{a-1} + \frac{a(a-1)}{2} c^{a-2} + \dots + \binom{a}{a-1} c\right]}_{\text{divisible by } c}$$

$a = 2k + 1$

$\downarrow$

b is divisible by 4 $\therefore$ let $b = 4m$.

$\therefore e = b+1 = 4m+1$.

$\therefore c-1 = 4m$

$a = 2k+1$

divisible by c

$\therefore a^{c-1} - 1 = a^{4m} - 1 = (a^2)^{2m} - 1 = (2c-1)^{2m} - 1$

$= (2c)^{2m} + 2m(2c)^{2m-1} + \dots + \binom{2m}{2m-1}(2c) + 1 - 1$

$= \underbrace{(2c)^{2m} + 2m(2c)^{2m-1} + \dots + \binom{2m}{2m-1}(2c)}$

$\hookrightarrow$ divisible by c

Let $a, b, c, d > 0$, be any real numbers. Then the maximum possible value of $cx + dy$, over all points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, must be

(a) $\sqrt{a^2c^2 + b^2d^2}$.

(b) $\sqrt{a^2b^2 + c^2d^2}$

(c) $\sqrt{\frac{a^2c^2+b^2d^2}{a^2+b^2}}$.

(d) $\sqrt{\frac{a^2b^2+c^2d^2}{c^2+d^2}}$.

$30 + 56 \le \sqrt{74} \times 10$.

$86 \le 8.7 \times 10$.

$5\times6 + 7\times8 \le \sqrt{5^2+7^2}\sqrt{6^2+8^2}$.

$pq + rs \le \sqrt{p^2+r^2}\sqrt{q^2+s^2}$

$cx + dy = ca\left(\frac{x}{a}\right) + db\left(\frac{y}{b}\right)$

$cx + dy \le \sqrt{(ca)^2 + (db)^2}\sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2}$

$cx + dy \le \sqrt{a^2c^2 + b^2d^2}$

Cauchy-Schwartz inequality

The value of

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+3+\dots 2021}$$

is

(a) $\frac{2021}{1010}$.

(b) $\frac{2021}{1011}$.

(c) $\frac{2021}{1012}$.

(d) $\frac{2021}{1013}$.

telescopic series

write each term as the difference of 2 terms

$1+2=3, \quad 1+2+3=6$

$t_n = \frac{1}{1+2+3+\dots+n}$

$= \frac{1}{} = \frac{2}{} = 2\left[\frac{1}{n} - \frac{1}{n+1}\right]$

(d) $\frac{2021}{1013}$.

$1+2+3+\cdots+n$

$= \frac{1}{n(n+1)} = \frac{2}{n(n+1)} = 2\left[\frac{1}{n} - \frac{1}{n+1}\right]$

$S_n = \sum t_n = 2\sum\left[\frac{1}{n} - \frac{1}{n+1}\right]$

$t_1 = 2\left[\frac{1}{1} - \cancel{\frac{1}{2}}\right]$

$t_2 = 2\left[\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right]$

$t_3 = 2\left[\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right]$

$\vdots$

$t_{2021} = 2\left[\cancel{\frac{1}{2021}} - \frac{1}{2022}\right]$

$S_{2021} = 2\left[1 - \frac{1}{2022}\right] = \frac{2\times 2021}{2022} = \frac{2021}{1011}$

Define $a = p^3 + p^2 + p + 11$ and $b = p^2 + 1$, where p is any prime number. Let $d = \gcd(a, b)$. Then the set of possible values of d is

$d = \text{Hcf}(a,b)$

(a) $\{1, 2, 5\}$.

(b) $\{2, 5, 10\}$.

(c) $\{1, 5, 10\}$.

(d) $\{1, 2, 10\}$.

$a = p^3+p^2+p+1+10.$

$a = (p^3+p) + (p^2+1) + 10.$

$a = p(p^2+1) + (p^2+1) + 10$

$a = (p^2+1)(p+1) + 10.$

$a = b(p+1) + 10.$

$a = d n_2 (p+1) + 10.$

$a = d n_1$

$b = d n_2.$

d has to divide both a and b

d has to be a divisor of 10.

$d = 1, 2, 5, 10$

If $d=1$ $a = n_1, b = n_2.$

$n_1 = n_2(p+1) + 10.$

$p = 3.$

$p+1 = 4.$

$a = n_1 = 25$

$b = n_2 = 5.$

$n_1 = 2r+1$

$n_2 = 2s+1$

$p \to$ even $p+1 \to$ odd.

$n_2 \to$ odd $n_1 \to$ odd

~~$p = 2k$~~

$n_1 = 4n_2 + 10.$

$2\ 5\ 10\ n_2 = 3, 5.$

$n_1 = n_2 \times 3 + 10.$

$p = 2$

$n_1 \to 19, 25$

$n_1 = 4n_2 + 10$

$n_2 = 1$

$n_1 = 14$

$n_2 \rightarrow 2, 5, 10$

$n_2 = 3, 5$

$n_1 = n_2 \times 3 + 10$

$p = 2$

$n_1 = (2s+1)3 + 10 = 6s + 13$

$\boxed{n_1 \rightarrow 19, 25}$