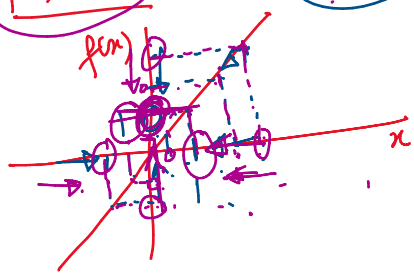
04 November 2023 17:00

Limits and continuity

Limit → an independent variable approaches a value, what is the value that the dependent variable will approach.

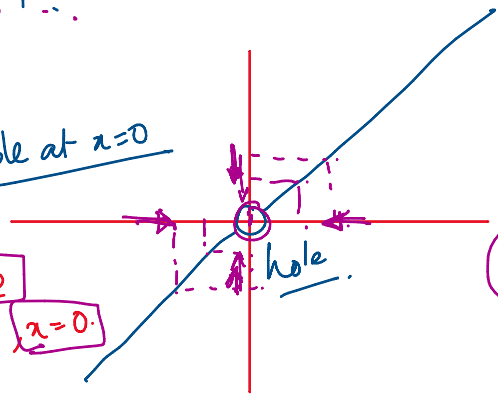
$f(x) = x$ $\quad x \to 1 \quad f(x) \to 1$.



Same or different?

$f(x) = \frac{x^2}{x}$ hole at $x=0$

$f(x) = x, \; x \neq 0$
$= \text{undefined}, \; x = 0$.



$x \to 0$, $f(x) \to 0$, $f(x)$ is approaching 0 but $f(x) \neq 0$

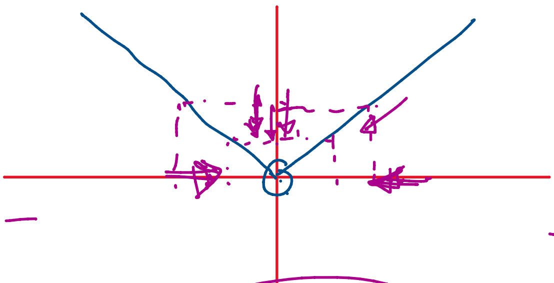
If $f(x)$ continues in its path then it will TEND towards 0.

$\lim\limits_{x \to 0} f(x) = 0$

as $x \to 0$, $f(x) \to 0$.

$f(x) = |x|$ $\qquad f(x) = x, \; x \geq 0$
$\qquad = -x, \; x < 0$.

when $x = 0$
$f(x) = x = 0$



$\lim\limits_{x \to 0_-} f(x) = \lim\limits_{x \to 0_-} (-x) = 0$

$\lim\limits_{x \to 0_+} f(x) = \lim\limits_{x \to 0_+} x = 0.$

$\lim\limits_{x \to a_-} f(x) = \lim\limits_{x \to a_+} f(x) = f(a)$ $\quad$ Limit exists for $x = 0$. $\quad$ $\lim\limits_{x \to 0_-} f(x) = \lim\limits_{x \to 0_+} f(x) = 0$.

⇓

$f(x)$ is continuous at $x = a$.

$f(x=0) = 0$ ✓. $\quad \lim\limits_{x \to 0_-} f(x) = \lim\limits_{x \to 0_+} f(x) = f(0)$

↓

$f(x)$ is CONTINUOUS at $x = 0$

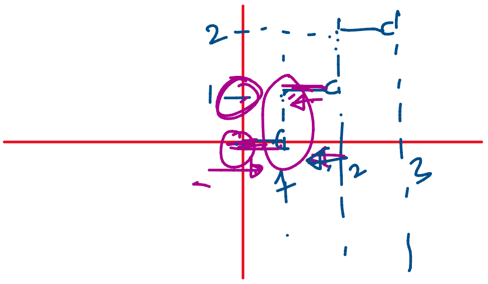
$f(x) = \text{gint}(x)$

$\lim\limits_{x \to 1} f(x)$

$0 \leq x < 1$
$\dots = 0$

$0 \le x < 1$
$\text{gint}(x) = 0$

$1 \le x < 2$
$\text{gint}(x) = 1$



$\lim_{x \to 1} f(x)$

$\lim_{x \to 1^+} f(x) = 1$

$\lim_{x \to 1^-} f(x) = 0$

$\lim_{x \to 1^+} f(x) \ne \lim_{x \to 1^-} f(x)$

$f(x)$ is discontinuous at $x = 1$

$\lim_{x \to 1} f(x)$ doesnt exist

$f(x) = \dfrac{x^2 - 5x + 6}{x^2 - 3x + 2}$ $\qquad \lim_{x \to 2} f(x) = ?$

$x = 2$ $\qquad \dfrac{0}{0}$ $\qquad f(x) = \dfrac{(x-2)(x-3)}{(x-2)(x-1)}$

$f(x) = \dfrac{x-3}{x-1}$ for all x except for $x = 2$

$\boxed{\text{discontinuity at } x = 2} \to \boxed{f(x) \text{ will have a hole at } x = 2}$

$\boxed{\text{removable discontinuity}}$

$f(x) = x$

$f(x) = \dfrac{x^2}{x}$

$f(x) = \dfrac{x-3}{x-1} \to$ Irremovable discontinuity

$\lim_{x \to 2^-} f(x) = \dfrac{2-3}{2-1} = -1$

$\lim_{x \to 2^+} f(x) = \dfrac{2-3}{2-1} = -1$

$\lim_{x \to 2} f(x)$ exists,

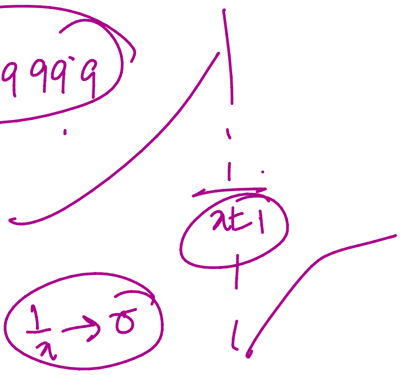
when $x \to 1$ $\quad f(x) \to \pm\infty$ $\qquad f(x) = \dfrac{x-3}{x-1}$

$\lim_{x \to 1^-} f(x) = +\infty$ $\qquad \dfrac{0.9-3}{0.9-1} = \dfrac{-2.1}{-0.1} = 21$

$\lim_{x \to 1^+} f(x) = -\infty.$

$\dfrac{1.1-3}{1.1-1} = \dfrac{-1.9}{0.1} = -19$ $\qquad \dfrac{0.9999-3}{0.9999-1} = \dfrac{-2.0001}{-0.0001} = 20001$

$\dfrac{1.0001-3}{1.0001-1} = \dfrac{-1.9999}{0.0001} = -19999$



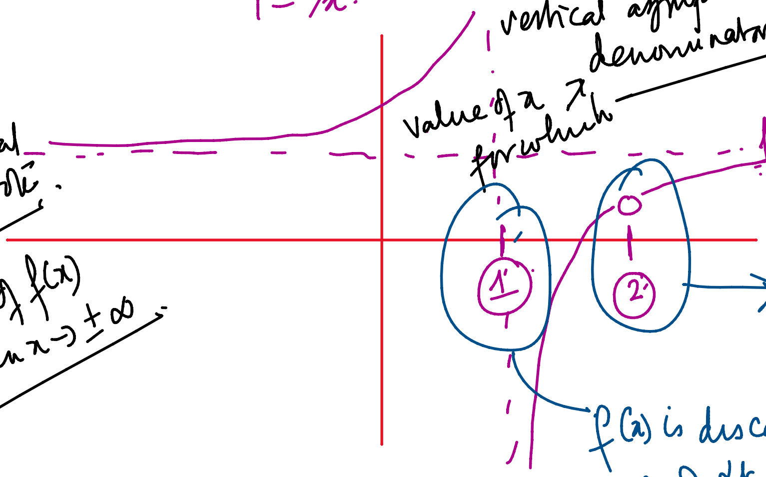
$f(x) = \dfrac{x-3}{x-1}$ $\qquad x \to +\infty$ $\qquad \dfrac{1}{x} \to 0$

$\lim f(x) = 1$ $\qquad \lim f(x) = 1$

$$= \frac{1-3/x}{1-1/x}$$

$\lim_{x\to+\infty} f(x) = 1$ $\qquad$ $\lim_{x\to-\infty} f(x) = 1$

$$f(x) = \frac{(x-2)(x-3)}{(x-2)(x-1)}$$



for. $f(x) = \dfrac{g(x)}{h(x)}$ where g(x) and h(x) are polynomials

Steps

① factorize g(x) and h(x)

If there is a common factor ⇒ hole ⇒ lt will exist but f(x) will be discontinuous (removable discontinuity)

② Cancel out the common factors.

⇒ If $f(x) = \dfrac{\phi(x)}{\psi(x)}$ → $\psi(x) = 0$ ⇒ horizontal asymptote.

③ divide the numerator ($\phi(x)$) and denominator ($\psi(x)$) by the highest power of x.

Eg $\dfrac{x^3+x^2+2}{3x^3+5x+1} = \dfrac{1+1/x+2/x^3}{3+5/x^2+1/x^3}$

for. $x \to \pm\infty$, $\frac{1}{x} \to 0$ → $\frac{1}{3}$ → horizontal asymptote.

$$f(x) = \frac{\cancel{(x+1)}(x-1)\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x+3)(x-4)\cancel{(x+1)}} = \frac{x^2-4x+3}{x^2-x-12} = \frac{1-4/x+3/x^2}{1-1/x-12/x^2}$$

$$f(x) = \frac{\ldots}{\cancel{(x-2)}(x+3)(x-4)\cancel{(x+1)}} = \frac{\ldots}{x^2 - x - 12} = \frac{\ldots}{1 - \frac{1}{x} - \frac{12}{x^2}}$$

how many holes? $\boxed{x = 2, -1}$

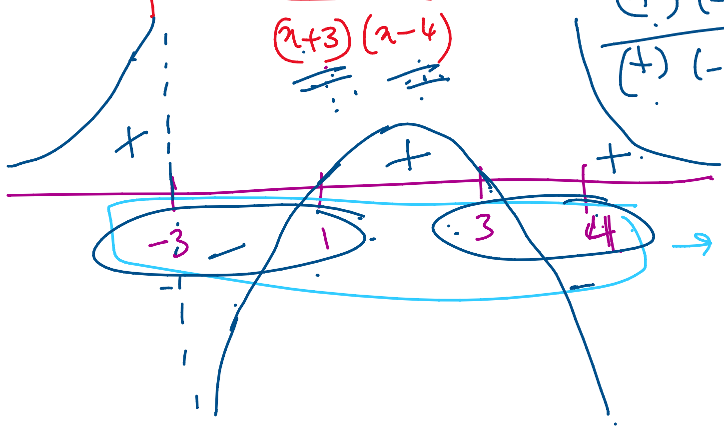
vertical asymptotes? $\boxed{x = -3, 4}$ $f(x) \to \pm\infty$

horizontal asymptote? $\boxed{y = 1}$

$f(x) > 0$ $f(x) < 0$

$$f(x) = \frac{(x-1)(x-3)}{(x+3)(x-4)}$$

$$\frac{(+)(-)}{(+)(-)} \qquad \frac{+\ +}{+\ -}$$



→ critical points