GROUP THEORY

Thursday, November 16, 2023 7:36 PM

Binary Composition

$S \to Set$

$(*)$ mapping $S \times S \to S$ is defined as $*(a,b)$

$a * b$

$5 \otimes 2$

$(2) \times (2) \to (4)$

$(+)$ $2+2=4$

$\sqrt{2}+\sqrt{3} \neq \sqrt{5}$ ✓

$a = 2+\sqrt{3}$ ✓

$b = -\sqrt{3}$ ✓

$a, b \in R \setminus Q$

$a+b = 2 \notin R \setminus Q$

Magmoid

(G) $(*)$ $(G, *)$ $(G, +)$

Semigroup

$(G, *)$

$a * (b * c) = (a * b) * c$

Monoid

$(G, *) \to (G, *)$

(i) ✓ $a * (b * c) = (a * b) * c$

(ii) ✓ $\exists\ e \in G$ such that $a * e = e * a = a$ $\forall\ a \in G$

Gmp

$(G, *)$

(i) ✓

(ii) ✓

(iii) $\forall\ a \in G \quad \exists\ a^{-1} \in G$ ✓

(ii)

(iii) $\forall a \in G \ \exists a^{-1} \in G$

$a * a^{-1} = a^{-1} * a = e$

Group

(i) $\mathbb{Z}$ → semigroup / [illegible]

$\mathbb{Z} + \mathbb{R}$ monoid → Semi . [illegible]

$\mathbb{Z} + \mathbb{R} + \mathbb{C}$ group / → Complete

Mahatma Gandhi → C + R

$2 \times 3 = 6$

$\{0\}$ → $n = 1$ $\quad 5 \times 0 = 0$

$\{\}$ $\quad n = 0$

Example

addition

addition

Matrix $\quad m \times n$

$\mathbb{Z}_3 = \{0, 1, 2\}$

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Hindu → Hindu

closure property

modulo

$\frac{7}{5}$ $\quad R = 2$

$\frac{a}{b}$ $\quad a < b$, $R = a$

$\frac{5}{7}$ $\quad R = 5$

$\{1, 2, 3, 4\}$

X_5

X_5	1	2	3	4

$\wedge_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

closure

$(a^{-1})^{-1} = a$

$a + b = b + a$

Ring

Empty set . $\begin{cases} + \\ \times \end{cases}$

Gandu ??

$a + b = b + a$

$(a + b) + c = (a) + (b + c)$

$a + 0 = a$

~~$a \times b$~~

Commutative Ring :-

$ab = ba$

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Field

A Commutative Ring

Unit ring →

If every non 0 element a unit

$G = \{5, 15, 25, 35\}$ X modulo of 40

Modulus $a \equiv b \pmod{n}$

Modulus $a \equiv b \pmod n$

Congruent

Remainder $\rightarrow -2$

$16 \equiv 1 \pmod 3$ $\rightarrow -1$

$14 \equiv 2 \pmod 3$

$30 \equiv 3 \pmod 3$

$30 \equiv 0 \pmod 3$

$1-(-2) = 3$

$2-(-1) = 3$

$a_1 \equiv b_1 \pmod c$

$a_2 \equiv b_2 \pmod c$

$a_1 + a_2 \equiv b_1 + b_2 \pmod c$

$2^{55}+1$ is it divisible by 11

$2^5 = 32 \equiv 10 \pmod{11}$

$32 \equiv -1 \pmod{11}$

$(2^5)^{11}$ 2^{55}

$(32)^{11} \equiv (-1)^{11} \pmod{11}$

$2^{55} = 32^{11} \equiv -1 \pmod{11}$

$2^{55}+1 \equiv -1+1 \pmod{11}$

$2^{55}+1 \equiv 0 \pmod{11}$

$2^{55}+1$

$\rightarrow$ 11 Answer

→ 11 Annulle ...

$G = \{5, 15, 25, 35\}$ mod 40

$\times_{40}$	5	15	25	35
5	25	35	5	15
15	35	25	15	5
25	5	15	25	35
35	15	5	35	25

$35 \times 20 = 700$

$40 \,/\, 525 \,/\, 13$

875

Identity → 25 both G

G form a Group

order (4) finite $a \in G$ $a = a^{-1}$

$a^2 = e$

$O(25) = 1$ (1 or 2)

exponent = (2)

Lagrange's Theorem: order of Subgroup of a finite Group divides the order of the Group.

H ∩ K

$(H \triangleleft K) \to G$ Find the order of the group.

$$O(HK) = \frac{O(H)\,O(K)}{O(H \cap K)}$$

Abelian Group

Commutative $ab = ba$ $\forall\ a, b \in G$

$(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$ $(\mathbb{R}, +)$ $(\mathbb{C}, +)$

$(\mathbb{Q}, \cdot)$ $(\mathbb{R}, \cdot)$ $(\mathbb{C}, \cdot)$

Normal Subgroup: A subgroup H of a Group G If $Ha = aH$ $\forall\ a \in G$

2011 G denotes a group 2×2 invertible matrix $(\det \neq 0)$

$H_1 = \{A \in G, \det(A) = 1\}$

$H_2 = \{A \in G,$ A is upper triangular$\}$

P: H_1 is a normal subgroup of G ✓

Q: H_2 is a normal subgroup of G ✗

a) P, Q True b) P ✓ Q ✗ c) P ✗ Q ✓ d) P ✗ Q ✗

i) $A, B \in H_1 \Rightarrow A, B \in G$ $|A| = 1$ $|B| = 1$

$|AB^{-1}| = |A||B^{-1}| = |A||B|^{-1}$

i) $A, B \in H_1 \Rightarrow A, B \in G$ $|A| = 1$ $|B| = 1$

$|AB^{-1}| = |A||B^{-1}| = |A||B|^{-1}$

$\Rightarrow |A| \cdot \frac{1}{|B|} = 1 \cdot 1 = 1$ $\left[\because |B^{-1}| = |B|^{-1}\right]$

H_1 is a subgroup of G

Let, $A \in H_1$, $P \in G$

$|A| = 1$, P is an invertible matrix, P^{-1} exists

$|P^{-1}AP| = |P^{-1}||A||P| = 1$ $\left[\because |A| = 1\right]$

$\therefore P^{-1}AP \in H_1 \ \forall A \in H_1$

$P \in G \Rightarrow H_1$ is a normal subgroup of G.

ii) $A, B \in H_2$

$A, B \in G$ such that A, B are U/T/M

B^{-1} is also an invertible U/T/M.

$AB^{-1} \rightarrow$ a U/T/M

$AB^{-1} \in H_2 \ \forall A, B \in H_2$ is a subgroup

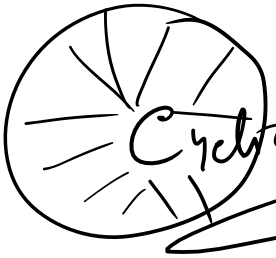
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$B^{-1}AB = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix} \notin H_2$$

$$= \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix} \notin H_2$$

H_2 is not a normal subgroup of G.



$b \in G$

$\Rightarrow$ every element of G is a power of b

$b \Rightarrow$ Generator of G $\quad \langle b \rangle$

$G = \langle b \rangle$ then $o(a) = o(b)$

eg: (X) $\{1, \omega, \omega^2\}$ (ω^3) $\rightarrow$ Generated by ω

(X) $\{1, -1, i, -i\}$ finite non group

$i = \sqrt{-1}$

$-1 \quad -i^2$

$(-1)(+1)$

$\Rightarrow (-1)$