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Determinants

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $\quad |A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

How to calculate det $|A|$?

$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

suppose we have 2×2 matrix

$B = \begin{bmatrix} 1 & 3 \\ 7 & 2 \end{bmatrix}_{2\times 2}$

$\therefore$ det B ie $|B| = \begin{vmatrix} 1 & 3 \\ 7 & 2 \end{vmatrix}$

$= (2 \times 1) - (7 \times 3)$

$= 2 - 21 = -19$

3x3 matrix : $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$|A| = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$

$= 1(45 - 48) - 2(36 - 42) + 3(32 - 35)$

$= -3 - 2(-6) + 3(-3)$

$= -3 + 12 - 9 = -15 + 12 = -3$ (ans).

Minors of determinants: (M_{ij})

say we have 2×2 matrix. $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Minors

$M_{11} = 4$ $\quad M_{21} = 3$

$M_{12} = 2$ $\quad M_{22} = 1$

Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 1 \\ -4 & 4 & 7 \end{bmatrix}$$

Let's write down the minors of A.

$$M_{11} = \begin{vmatrix} 5 & 1 \\ 4 & 7 \end{vmatrix} = 35 - 4 = 31$$

$$M_{12} = \begin{vmatrix} 3 & 1 \\ -4 & 7 \end{vmatrix} = 21 + 4 = 25$$

$$M_{13} = \begin{vmatrix} 3 & 5 \\ -4 & 4 \end{vmatrix} = 12 - (-20) = 32$$

$$M_{21} = \begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} = 14 - 12 = 2$$

$$M_{22} = \begin{vmatrix} 1 & 3 \\ -4 & 7 \end{vmatrix} = 7 + 12 = 19$$

$$M_{23} = \begin{vmatrix} 1 & 2 \\ -4 & 4 \end{vmatrix} = 4 - (-8) = 12$$

$$M_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = 2 - 15 = -13$$

$$M_{32} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 1 - 9 = -8$$

$$\begin{aligned} |A| &= 1(35-4) \\ &\quad -2(21+4) \\ &\quad +3(12+20) \\ &= 31 - 50 + 96 \\ &= 127 - 50 \\ &= 77 \text{ (ans)}. \end{aligned}$$

$$M_{32} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 1-9 = -8$$

$$m_{33} = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 5-6 = -1$$

Cofactors : The cofactors C_{ij} of $A = [a_{ij}]$ is equal to $(-1)^{i+j}$ times the det of submatrix (minor) of order $(n-1)$ obtained by leaving i^{th} row and j^{th} column of A.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 1 \\ 4 & 4 & 7 \end{bmatrix}_{3\times3}$ ↳ 9 cofactors.

$C_{ij} = (-1)^{i+j}$

$C_{11} = +$ $\quad C_{12} = -$ $\quad C_{13} = +$

$C_{21} = -$ $\quad C_{22} = +$ $\quad C_{23} = -$

$C_{31} = +$ $\quad C_{32} = -$ $\quad C_{33} = +$

Cofactors are: $C_{11} = (-1)^{1+1} M_{11}$

$$= 1 \cdot \begin{vmatrix} 5 & 1 \\ 4 & 7 \end{vmatrix}$$

$$= 1 \cdot (35-4)$$

$$= 31$$

$$C_{12} = (-1)^{1+2} M_{12}$$

$$= -1 \cdot \begin{vmatrix} 3 & 1 \\ -4 & 7 \end{vmatrix}$$

$$= -25$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 5 \\ -4 & 4 \end{vmatrix} = 12-(-20)$$

$$= +32$$

$$C_{21} = (-1)^{2+1} \cdot M_{21}$$

$$= -1 \cdot \begin{vmatrix} 2 & 3 \end{vmatrix}$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \end{vmatrix}$$

$C_{21} = (-1)^{2+1} M_{21}$

$= -1 \cdot \begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} = -2$

$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ -4 & 4 \end{vmatrix}$

$= -12$

$C_{22} = (-1)^{2+2} M_{22}$

$= 1 \begin{vmatrix} 1 & 3 \\ -4 & -7 \end{vmatrix} = 19$

$C_{31} = (-1)^{3+1} M_{31}$

$= +1 \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix}$

$= 3 - 15$

$= -13$

$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}$

$= -1\,(1-9) = -(-8) = 8$

$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}$

$= 1 \cdot (5-6) = -1$

Solving simultaneous eqn using **Cramer's Rule:**

Suppose we have the following system of linear equations:

$$a_1 x + b_1 y + c_1 z = k_1$$
$$a_2 x + b_2 y + c_2 z = k_2$$
$$a_3 x + b_3 y + c_3 z = k_3$$

we can write the system of equations in matrix form as:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$$\begin{bmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} k_2 \\ k_3 \end{bmatrix}$$

Here,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

$$\Delta_x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

$$\Delta_y = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix} \neq 0$$

$$\Delta_z = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix} \neq 0$$

Thus, the solution of system of equation is given by

$$x = \frac{\Delta_x}{\Delta} \; ; \; y = \frac{\Delta_y}{\Delta} \; ; \; z = \frac{\Delta_z}{\Delta}$$

↳ This is Cramer's Rule.

Q Solve the following using Cramer's Rule:

$$5x - 7y + z = 11$$
$$6x - 8y - z = 15$$
$$3x + 2y - 6z = 7$$

$$\Delta = \begin{vmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{vmatrix} = 5(48+2) - (-7)(-36+3) + 1(12+24)$$

$$\begin{vmatrix} 6 & -8 & -1 \\ 3 & 2 & -6 \end{vmatrix} \quad + 1(12+24)$$

$$= 5\times 50 + 7\times -33 + 36$$

$$= 250 - 231 + 36$$

$$= 55$$

$$\Delta_x = \begin{vmatrix} 11 & -7 & 1 \\ 15 & -8 & -1 \\ 7 & 2 & -6 \end{vmatrix}$$

$$= 11(48+2) + 7(-90+7) + 1(30+56)$$

$$= 55$$

$$\Delta_y = \begin{vmatrix} 5 & 11 & 1 \\ 6 & 15 & -1 \\ 3 & 7 & -6 \end{vmatrix} = 5(-90+7) - 11(-36+3) + 1(42-45)$$

$$= -55$$

$$\Delta_z = \begin{vmatrix} 5 & -7 & 11 \\ 6 & -8 & 15 \\ 3 & 2 & 7 \end{vmatrix} = 5(-56-30) + 7(42-45) + 11(12+24)$$

$$= -55$$

∴ By using Cramer's Rule

$$x = \frac{\Delta_x}{\Delta} = \frac{55}{55} = 1 \quad ; \quad y = \frac{\Delta_y}{\Delta} = \frac{-55}{55} = -1$$

and $z = \frac{\Delta_z}{\Delta} = \frac{-55}{55} = -1$

∴ value of x, y and z are 1, −1 and −1 resp. (ans).

Solve: $\left.\begin{aligned} 4x - 2y + 3z &= 3 \\ 6x - 3y + 3z &= 5 \end{aligned}\right\}$ solve x, y, z using cramer's rule.

Also find cofactors of determinant.