Linear Transformation:

$T: U \to V \quad T(c_1 \underset{\sim}{\alpha} + c_2 \underset{\sim}{\beta}) = c_1 T(\underset{\sim}{\alpha}) + c_2 T(\underset{\sim}{\beta}) \Rightarrow T$ is a linear transformatio

(i) $N(T) = \{ \underset{\sim}{\alpha} \in U : T(\underset{\sim}{\alpha}) = 0 \}$.

(ii) Nullity: Dimension of $N(T)$

(iii) $R(T) = \{ \underset{\sim}{v} \in V \text{ s.t } T(\underset{\sim}{u}) = \underset{\sim}{v}, \ \underset{\sim}{u} \in U \}$.

(iv) Rank of T : Dimension of $R(T)$

Rank Nullity Theorem: Rank of T + Nullity of T = Dim of U.

Q. Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation:

$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_2)$. Find Rank of T.

$U = \mathbb{R}^2 \Rightarrow$ Dim of $U = 2 \Rightarrow$ Rank of T = Dim of U − Nullity of T.

$= 2 -$ Nullity of T

$= 2 - 1 = 1$.

Defn: $N(T) = \{ \underset{\sim}{\alpha} \in \mathbb{R}^2 : T(\underset{\sim}{\alpha}) = \underset{\sim}{0} \}$.

Consider $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$.

$$T(\underset{\sim}{\alpha}) = 0 \Rightarrow T(\alpha_1, \alpha_2) = 0.$$

$$(\alpha_1 + \alpha_2, \alpha_1 - \alpha_2, \alpha_2) = (0, 0, 0)$$

$$\left.\begin{aligned} \alpha_1 + \alpha_2 &= 0 \\ \alpha_1 - \alpha_2 &= 0 \\ \alpha_2 &= 0 \end{aligned}\right\} \Rightarrow \alpha_1 = 0, \ \alpha_2 = 0 \qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$N(T) = \{ (0, 0) \} \Rightarrow$ Nullity $= 1$.

Note: For a matrix $A_{m \times n}$:

Null space of matrix A: $N(A) = \{ \underset{\sim}{x} : A\underset{\sim}{x} = \underset{\sim}{0} \}$.

System of homogeneous equations.

Interpretation: $N(A)$ is the set of all solutions to the system of homogeneous equations.

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Nullity of A: Dimension of $N(A)$.

Rank-Nullity Th: Rank of A + Nullity of A = No. of cols of A.

Recap: Rank of a matrix $A_{m \times n}$:

Defn: No. of independent rows/columns in matrix.

Properties:

(i) $R(A_{m \times n}) \leq \min\{m, n\}$.

(ii) $R(AB) \leq \min\{R(A), R(B)\}$. $\quad A_{m \times n} B_{n \times p}$.

(iii) $R(A) = R(A^T)$

Q. $A = \begin{bmatrix} \boxed{1} & 2 & 2 & -1 & 2 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 4 \end{bmatrix}_{3 \times 5}$ $\qquad R(A) \leq 3$

To find $R(A)$ we use elementary row operations:

(i) Check first non-zero element in R_1. Use that to make all the elements below it zero.

$$\xrightarrow[R_3' \to R_3 - 3R_1]{R_2' \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 2 & -1 & 2 \\ 0 & 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & 2 & 4 & -2 \end{bmatrix}$$

(ii) Check the first non-zero element in R_2, use that to make all the elements below it zero.

$$\xrightarrow{R_3' \to R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 2 & -1 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow R(A) = 2.$$

Q. $A = \begin{bmatrix} \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}_{4 \times 4}$ $\qquad R(A) \leq 4$

$$\xrightarrow{R_3' \to R_3 - R_1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3' \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & \boxed{-1} & 0 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4' \to R_3 + R_4} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow R(A) = 3.$$

Q. Consider the system as:

$$x_1 + 2x_2 + 2x_3 - x_4 + 2x_5 = 0.$$
$$x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 0.$$
$$3x_1 + 6x_2 + 8x_3 + x_4 + 4x_5 = 0$$

Find the dimension of the solution space.

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 & 2 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 4 \end{bmatrix}_{3\times 5}.$$

$$S = \underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$S = \{ \underset{\sim}{x} : A\underset{\sim}{x} = \underset{\sim}{0} \} = N(A)$$

Dim of soln space = Nullity of A.

Rank of A + Nullity of A = No. of cols.

Nullity of A = 5 − 2 = 3