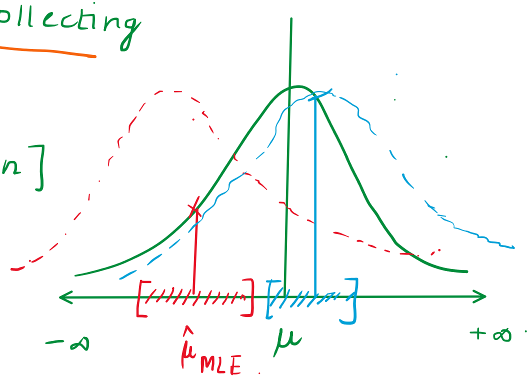
Bayesian Inference

Classical Inference [Axiomatic Inference] vs Bayesian Inference.

Recall: $X_1, X_2, \ldots, X_n \overset{iid}{\sim} f_\theta(x)$ [θ = unknown popln parameter].

Classical: Assume that θ is a constant given value (unknown). Estimate/Infer about this fixed value based on sample obs, using different techniques.

Bayesian: Assume that θ is a random variable. Begin with assuming that before collecting any sample obs, we have some distn of θ, say $\pi(\theta)$. [Prior distn]



Eg: $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$

[θ = unknown, σ^2 = known].

(i) Begin with a prior distn of θ, say $\theta \sim \pi(\theta) = N(\mu, \tau^2)$ (μ, τ^2 are known), μ = prior belief about the unknown popln parameter, τ^2 = variance of prior mean indicates the strength of our belief in μ. Larger $\tau^2 \Rightarrow$ less sure are we about μ being the exp value of θ.

Alternatively, $\theta \sim \pi(\theta) = \text{Uniform}[a, b]$.

$\therefore$ Choice of Prior distribution is "subjective".

$\therefore$ Prior distn $\pi(\theta)$: Prob distn of the unknown popln parameter that captures the initial belief about θ.

Let Ω: Parameter space.

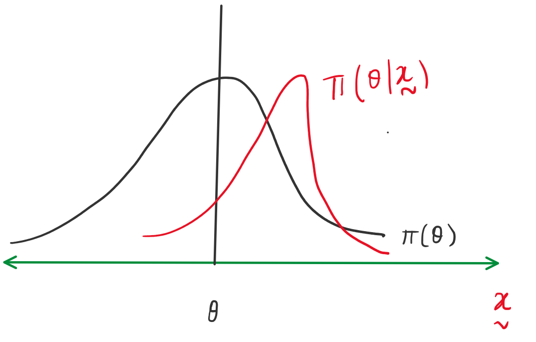
∴ If θ is a discrete r.v : condition for $\pi(\theta)$ to be a valid pmf will $\sum_{\theta \in \Omega} \pi(\theta) = 1$.

∴ If θ is a continuous r.v : condition for $\pi(\theta)$ to be a valid pdf will $\int_{\theta \in \Omega} \pi(\theta) = 1$.

(ii) Let $\underset{\sim}{x} = \{x_1, x_2, \ldots, x_n\}$ set of observed sample values.

Once the sample data is collected from the popln, we "update" our belief about θ. This updated belief is captured the "posterior distn" of θ. Denote as $\boxed{\pi(\theta | \underset{\sim}{x})}$

Posterior distn $\pi(\theta | \underset{\sim}{x})$ is used in Bayesian inference about θ.



Define: The posterior distn as follows:

Case I: If θ is discrete r.v.

$$\pi(\theta | \underset{\sim}{x}) = \frac{f(\underset{\sim}{x} | \theta) \cdot \pi(\theta)}{\sum_{\theta \in \Omega} f(\underset{\sim}{x} | \theta) \cdot \pi(\theta)}$$

Case II: If θ is a continuous r.v

$$\pi(\theta | \underset{\sim}{x}) = \frac{f(\underset{\sim}{x} | \theta) \cdot \pi(\theta)}{\underbrace{\int_{\theta \in \Omega} f(\underset{\sim}{x} | \theta) \cdot \pi(\theta)}_{\text{normalizing constant.}}}$$

For $\pi(\theta | \underset{\sim}{x})$ to be a valid pmf:

$$\sum_{\theta \in \Omega} \pi(\theta | \underset{\sim}{x}) = 1.$$

For $\pi(\theta | \underset{\sim}{x})$ to be a valid pdf:

$$\int_{\theta \in \Omega} \pi(\theta | \underset{\sim}{x}) = 1$$

Check: LHS: $\int_{\theta \in \Omega} \pi(\theta | \underset{\sim}{x})$

$$= \int_{\theta \in \Omega} \frac{f(\underset{\sim}{x} | \theta) \cdot \pi(\theta)}{\underbrace{\int_{\theta \in \Omega} f(\underset{\sim}{x} | \theta) \cdot \pi(\theta) \, d\theta}_{\text{constant}}} \, d\theta$$

$$= \int_{\theta \in \Omega} \frac{\int_{\theta \in \Omega} f(\underset{\sim}{x}|\theta) \cdot \pi(\theta)\, d\theta}{}$$

constant

$$= \frac{\int f(\underset{\sim}{x}|\theta) \cdot \pi(\theta) \cdot d\theta}{\int f(\underset{\sim}{x}|\theta) \cdot \pi(\theta) \cdot d\theta} = 1.$$

(Proved)

Eg: $X \sim Poi(\lambda)$.

$$f(x) = e^{-\lambda} \boxed{\frac{\lambda^x}{x!}}$$

Eg: $X \sim N(\mu, \sigma^2)$.

$$\underline{f(x)} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

HW. $\pi(\theta|\underset{\sim}{x}) = f(\underset{\sim}{x}|\theta) \cdot \pi(\theta)$.

$\therefore$ To make it a valid pdf: $\pi(\theta|\underset{\sim}{x}) = c \cdot f(\underset{\sim}{x}|\theta)\, \pi(\theta)$.

To find 'c', $\int \pi(\theta|\underset{\sim}{x})\, d\theta = 1$.