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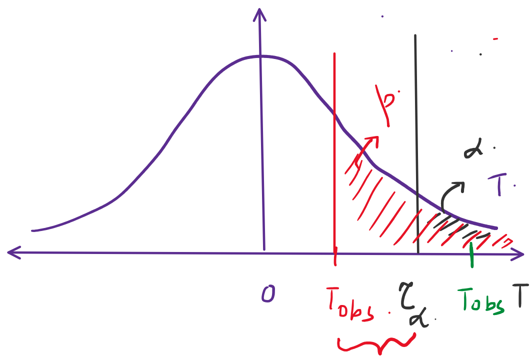
p-value: the probability that the test statistic will take a value more extreme than the one already observed, given H_0 is true.

Eg: To test $H_0: \mu = \mu_0$ vs $H_1: \mu > \mu_0$.

Test statistic $T = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}} \overset{H_0}{\sim} N(0,1)$.



Fix the level of significance $= \alpha$

$(0.01/0.05)$

If $T_{obs} > \tau_\alpha \Rightarrow$ Reject H_0.

$(p \gtrless \alpha)$?

p-value $= P[T > T_{obs}]$.

If $p > \alpha \Rightarrow T \in W^c \Rightarrow$ Accept H_0

If $p < \alpha \Rightarrow T \in W \Rightarrow$ Reject H_0

Conclusion:

(i) Based on critical value: If $T_{obs} > \tau_\alpha \Rightarrow$ Reject H_0

(ii) Based on p-value: $p_{obs} < \alpha \Rightarrow$ Reject H_0.

Testing for Mean of a Normal Population:-

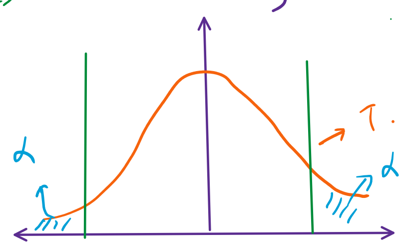
Let us consider a r.s $X_1, X_2, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$.

Case I: σ is known.

To test $H_0: \mu = \mu_0$ vs

$H_{1A}: \mu > \mu_0 \Rightarrow$ Right tail.
$H_{1B}: \mu < \mu_0 \Rightarrow$ Left tail.
$H_{1C}: \mu \neq \mu_0 \Rightarrow$ Both tail

Test statistic $T = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \overset{H_0}{\sim} N(0,1)$.

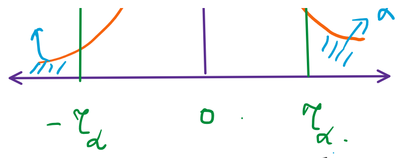


Fix L.O.S $= \alpha$.

Fix L.O.S = α.

(i) If $T_{obs} > \tau_\alpha \Rightarrow$ Reject H_0 at α% L.O.S.

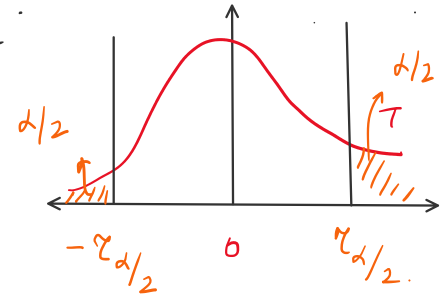
(ii) If $T_{obs} < -\tau_\alpha \Rightarrow$ Reject H_0 at α% L.O.S.



$\alpha + \alpha = 2\alpha$.

(iii) Reject H_0, if $T_{obs} > \tau_{\alpha/2}$ or $T_{obs} < -\tau_{\alpha/2}$.
at α% L.O.S.

Combining: $|T_{obs}| > \tau_{\alpha/2}$.



Case II: If σ is unknown.

Note: $T = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ [Cannot be computable as σ is unknown].

Replacing σ by its unbiased estimator.

Note: Suppose we have a popln parameter θ and we want to estimate it using an estimator T.
Then T is an unbiased estimator of θ ie $E(T) = \theta$.

Eg: If $X_1, X_2, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ then $E(\bar{X}) = \mu$.

$$E(\bar{X}) = E\left[\frac{1}{n}\sum X_i\right] = \frac{1}{n}\sum E(X_i) = \frac{1}{n}\sum \mu = \frac{1}{n} \cdot n\mu = \mu.$$

Define: $s'^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$, then $E(s'^2) = \sigma^2$.

$$E(s'^2) = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]$$

$$= E\left[\frac{n}{n-1} \cdot \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]$$

$$= \frac{n}{n-1} E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]$$

$$= \frac{n}{n-1} E\left[\frac{1}{n}\sum x_i^2 - \bar{x}^2\right]$$

$$= \frac{n}{n-1} E\left[\frac{1}{n}\sum x_i^2 - \bar{x}^2\right]$$

$$= \frac{n}{n-1}\left[\frac{1}{n}\sum \underbrace{E(x_i^2)}_{(\sigma^2+\mu^2)} - E(\bar{x}^2)\right].$$

Now, $X_i \sim N(\mu, \sigma^2)$: $E(x_i) = \mu$, $Var(x_i) = \sigma^2$.

$$E(x_i^2) - [E(x_i)]^2 = \sigma^2$$

$$E(x_i^2) = \sigma^2 + \mu^2$$

Also, $\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$: $E(\bar{x}) = \mu$, $Var(\bar{x}) = \frac{\sigma^2}{n}$

$$E(\bar{x}^2) - [E(\bar{x})]^2 = \frac{\sigma^2}{n}.$$

$$E(\bar{x}^2) = \frac{\sigma^2}{n} + \mu^2$$

$$= \frac{n}{n-1}\left[\frac{1}{n}\sum(\sigma^2+\mu^2) - \left(\frac{\sigma^2}{n}+\mu^2\right)\right]$$

$$= \frac{n}{n-1}\left[\sigma^2 + \cancel{\mu^2} - \frac{\sigma^2}{n} - \cancel{\mu^2}\right]$$

$$= \frac{n}{n-1}\left[\left(1-\frac{1}{n}\right)\sigma^2\right] = \frac{n}{n-1}\cdot\frac{n-1}{n}\cdot\sigma^2 = \sigma^2.$$

$\therefore$ On replacing σ by its unbiased estimator, new test statistic $T = \dfrac{\bar{x}-\mu}{s'/\sqrt{n}} \overset{H_0}{\sim} N(0,1)$ [No!]

$$T = \frac{\bar{x}-\mu}{s'/\sqrt{n}} \overset{H_0}{\sim} t_{(n-1)}$$

$$t_{(n)} = \frac{\tau}{\sqrt{\chi^2_{(n)}/n}}.$$

$$t_{(n-1)} = \frac{\tau}{\sqrt{\chi^2_{(n-1)}/(n-1)}}$$

We know, $\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$.

$$\tau \doteq \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1).$$

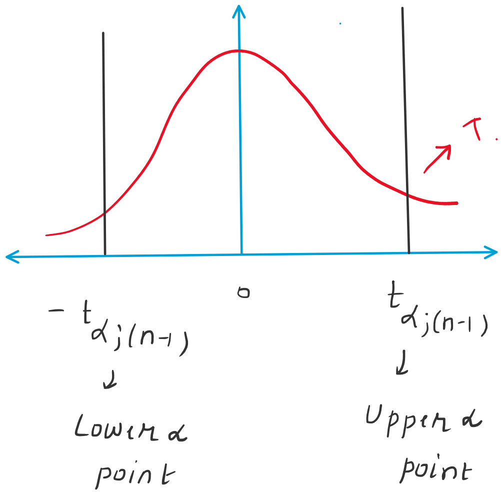
$$\therefore \frac{(\bar{x}-\mu)/(\sigma/\sqrt{n})}{\sqrt{\sum\left(\frac{x_i-\bar{x}}{\sigma}\right)^2/(n-1)}} \sim t_{(n-1)}$$

$$\sum\left(\frac{x_i-\bar{x}}{\sigma}\right)^2 \sim \chi^2_{(n-1)}$$

$$\sqrt{\sum\left(\frac{x_i-\bar{x}}{\sigma}\right)^2 \Big/ (n-1)} \qquad \Bigg| \; \sum\left(\frac{x_i-\bar{x}}{\sigma}\right)^2 \sim \chi^2_{(n-1)}$$

$$= \frac{\frac{1}{\sigma}\sqrt{n}(\bar{x}-\mu)}{\frac{1}{\sigma}\sqrt{\frac{1}{n-1}\sum(x_i-\bar{x})^2}} = \frac{\sqrt{n}(\bar{x}-\mu)}{s'} = \frac{\bar{x}-\mu}{s'/\sqrt{n}}.$$

$$T = \frac{\bar{x}-\mu}{s'/\sqrt{n}} \overset{H_0}{\sim} t_{(n-1)}$$



(i) We reject H_0, if $T_{obs} > t_{\alpha;(n-1)}$ at $\alpha\%$ L.O.S.

(ii) We reject H_0, if $T_{obs} < -t_{\alpha;(n-1)}$ at $\alpha\%$ L.O.S.

(iii) We reject H_0, if $T_{obs} > t_{\alpha/2;(n-1)}$ or $T_{obs} < -t_{\alpha/2;(n-1)}$ at $\alpha\%$ L.O.S.

$\therefore$ Combining $|T_{obs}| > t_{\alpha/2;(n-1)}$