HW.

Q. Use the NP lemma to find the best critical region to test : $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$ $[\theta_1 > \theta_0]$ for $N(\theta, \sigma^2)$ popln where σ^2 is known. Hence find the power of the test.

r.s $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ $[\sigma^2$ is known$]$.

$$f(x_i, \theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i - \theta}{\sigma}\right)^2}$$

$$\therefore L(x, \theta) = \prod_{i=1}^{n} f(x_i, \theta) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\Sigma(x_i - \theta)^2}.$$

(BCR)

NP: $\frac{L(x, \theta_1)}{L(x, \theta_0)} \geq K \quad \forall x \in W$ solve this to obtain BCR.

$$L(x, \theta) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2}\Sigma(x_i - \theta)^2}$$

$$H_0: L(x, \theta_0) = \frac{1}{(\cancel{\sigma\sqrt{2\pi}})^n} e^{-\frac{1}{2}\Sigma(x_i - \theta_0)^2}$$

$$H_1: L(x, \theta_1) = \frac{1}{(\cancel{\sigma\sqrt{2\pi}})^n} e^{-\frac{1}{2}\Sigma(x_i - \theta_1)^2}$$

$$\Rightarrow \frac{e^{-\frac{1}{2\sigma^2}\Sigma(x_i - \theta_1)^2}}{e^{-\frac{1}{2\sigma^2}\Sigma(x_i - \theta_0)^2}} \geq K.$$

$$\Rightarrow \exp\left[-\frac{1}{2\sigma^2}\left\{\Sigma(x_i - \theta_1)^2 - \Sigma(x_i - \theta_0)^2\right\}\right] \geq k.$$

$$\Rightarrow -\frac{1}{2\sigma^2}\left\{\Sigma(x_i - \theta_1)^2 - \Sigma(x_i - \theta_0)^2\right\} \geq \ln k$$

$$\Rightarrow -\frac{1}{2\sigma^2}\left\{\cancel{\Sigma x_i^2} + n\theta_1^2 - 2\theta_1\Sigma x_i - \cancel{\Sigma x_i^2} - n\theta_0^2 + 2\theta_0\Sigma x_i\right\} \geq \ln k$$

$$\Rightarrow -\frac{1}{2\sigma^2}\left\{n(\theta_1^2 - \theta_0^2) - 2\Sigma x_i(\theta_1 - \theta_0)\right\} \geq \ln k.$$

$$\Rightarrow \frac{1n}{\sigma^2}(\theta_1 - \theta_0)\frac{\Sigma x_i}{n} - \frac{n}{2\sigma^2}(\theta_1^2 - \theta_0^2) \geq \ln k.$$

$$\Rightarrow \frac{n}{\sigma^2}(\theta_1 - \theta_0)\bar{x} - \frac{n}{2\sigma^2}(\theta_1^2 - \theta_0^2) \geq \ln k.$$

$$\Rightarrow \frac{n}{\sigma^2}(\theta_1-\theta_0)\bar{x} - \frac{n}{2\sigma^2}(\theta_1^2-\theta_0^2) \geq \ln k$$

$$\Rightarrow \underbrace{(\theta_1-\theta_0)}_{>0}\bar{x} - \frac{\overbrace{\theta_1^2-\theta_0^2}^{>0}}{2} \geq \frac{\sigma^2}{n}\ln k .$$

$$\Rightarrow (\theta_1-\theta_0)\bar{x} \geq \frac{\sigma^2}{n}\cdot\ln k + \frac{\theta_1^2-\theta_0^2}{2} .$$

$$\Rightarrow \bar{x} \geq \underbrace{\frac{\sigma^2}{n}\cdot\frac{\ln k}{(\theta_1-\theta_0)} + \left(\frac{\theta_1+\theta_0}{2}\right)}_{\text{Denote } \lambda_1}$$

→ calculated based on sample obs & other known values.

To test: $H_0: \theta=\theta_0$ vs $H_1: \theta=\theta_1$ $[\theta_1>\theta_0]$

$\therefore$ BCR $= \{x \mid \bar{x} \geq \lambda_1\}$.

We know: $\alpha = P$ [Reject H_0 when H_0 is true] .

fixed earlier. ← $\alpha = P[\bar{x} \geq \lambda_1 \mid H_0: \theta=\theta_0]$.

r.s: $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$

$\bar{X} \sim N\left(\theta, \frac{\sigma^2}{n}\right)$ Under H_0: $\bar{X} \sim N\left(\theta_0, \frac{\sigma^2}{n}\right)$.

$$\alpha = P[\bar{x} \geq \lambda_1 \mid \theta=\theta_0]$$

$$\alpha = P\left[\left(\frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}}\right) \geq \frac{\lambda_1-\theta_0}{\sigma/\sqrt{n}}\right]$$

$$\underset{0.05}{\alpha} = P\left[Z \geq \frac{\lambda_1-\theta_0}{\sigma/\sqrt{n}}\right]$$

→ critical value say z_α. (obtained from standard normal tables).

$$z_\alpha = \frac{\lambda_1-\theta_0}{\sigma/\sqrt{n}} \Rightarrow \lambda_1 = \theta_0 + \frac{\sigma}{\ldots}\cdot z\ldots$$

...rmal tables).

$$Z_\alpha = \frac{\lambda_1 - \theta_0}{\sigma/\sqrt{n}} \Rightarrow \boxed{\lambda_1 = \theta_0 + \frac{\sigma}{\sqrt{n}} \cdot Z_\alpha}$$

Power of test: $1-\beta = P[x \in W \mid H_1] = P[\bar{x} \geq \lambda_1 \mid \theta = \theta_1]$.

$$= P\left[\left(\frac{\bar{x} - \theta_1}{\sigma/\sqrt{n}}\right) \geq \frac{\lambda_1 - \theta_1}{\sigma/\sqrt{n}}\right]$$

Under H_1, $\bar{X} \sim N\left(\theta_1, \frac{\sigma^2}{n}\right)$

$$= P\left[Z \geq \left(\frac{\lambda_1 - \theta_1}{\sigma/\sqrt{n}}\right)\right]$$

$$= 1 - \Phi\left(\frac{\lambda_1 - \theta_1}{\sigma/\sqrt{n}}\right)$$

Q. R.S $X_1, X_2, \dots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ where σ^2 is known.

To test $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$ $[\theta_1 < \theta_0]$. Find the BCR

BCR:

$$\frac{n}{\sigma^2}\underbrace{(\theta_1 - \theta_0)}_{<0}\bar{x} - \frac{n}{2\sigma^2}(\theta_1^2 - \theta_0^2) \geq \ln k$$

$$(\theta_1 - \theta_0)\bar{x} - \frac{\theta_1^2 - \theta_0^2}{2} \geq \frac{\sigma^2}{n} \cdot \ln k$$

$$\underbrace{(\theta_1 - \theta_0)}_{<0}\bar{x} \geq \frac{\sigma^2}{n} \ln k + \frac{\theta_1^2 - \theta_0^2}{2}$$

$$\bar{x} \leq \underbrace{\left(\frac{\sigma^2}{n} \cdot \frac{\ln k}{(\theta_1 - \theta_0)} + \left(\frac{\theta_1 + \theta_0}{2}\right)\right)}_{\lambda_2}$$

$$\bar{x} \leq \lambda_2.$$

$$\therefore \text{BCR} = \{x \mid \bar{x} \leq \lambda_2\}.$$

To test: $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$ $[\theta_1 > \theta_0]$ — $\theta > \theta_0$.

To test: $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$ $[\theta_1 < \theta_0]$ — $\theta < \theta_0$.

To test: $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$ $[\theta_1 > \theta_0]$

$$BCR = \{ x \mid \bar{x} \geq \lambda_1 \}.$$

To test: $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$ $[\theta_1 < \theta_0]$

$$BCR = \{ x \mid \bar{x} \leq \lambda_2 \}.$$

$\Rightarrow$ This test does not have a Uniformly Most Powerful (UMP) Critical Region.

UMP Test:

The region w is called UMP critical region for testing the hypothesis: $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$.

if $P(x \in w \mid H_1) \geq P(x \geq w_1 \mid H_1)$ $\forall$ w_1 and $\forall$ $\theta \neq \theta_0$.