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Lagrange Multiplier (λ)

[Constrained Optimisation (with two variables)]

$\bar{g} = g(x_1, x_2)$

$\bar{g} - g(x_1, x_2) = 0$

Max: $\boxed{U = x_1 \cdot x_2}$

s/t $\boxed{5 = 2x_1 + x_2}$

$5 - 2x_1 - x_2 = 0$

↓

Maximise or Minimise our objective function for max $y = f(x_1, x_2)$

s/to $\bar{g} = g(x_1, x_2)$

$$\boxed{L} = f(x_1, x_2) + \lambda(\bar{g} - g(x_1, x_2))$$

x_1, x_2, λ

F.O.C $\frac{\partial L}{\partial x_1} = 0$ $\quad \frac{\partial L}{\partial x_2} = 0$ $\quad \frac{\partial L}{\partial \lambda} = 0$

S.O.C (Bordered-Hessian determinant).

$$|H| = \begin{vmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial \lambda} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial \lambda} \\ \frac{\partial^2 L}{\partial \lambda \partial x_1} & \frac{\partial^2 L}{\partial \lambda \partial x_2} & \frac{\partial^2 L}{\partial \lambda^2} \end{vmatrix}_{3 \times 3}$$

$|H| > 0 \Rightarrow$ maximisation.

$|H| < 0 \Rightarrow$ minimisation.

Q optimize the utility function

$$U = 4xy - y^2$$

subject to ... constraint

$U = 4xy - y$

Subject to budget constraint

$2x + y = 6$

Objective is to max $U = 4xy - y^2$

subj to : $2x + y = 6$

Let us write the Lagrangian expression as,

$$L = (4xy - y^2) + \lambda(2x + y - 6)$$

For maximisation F.O.C requires,

$\frac{\partial L}{\partial x} = 0$

or, $4y + 2\lambda = 0$

or, $\lambda = -\frac{4y}{2} = -2y$ — (1)

$\frac{\partial L}{\partial y} = 0$

or, $4x - 2y + \lambda = 0$

$\lambda = 2y - 4x$ — (2)

$\frac{\partial L}{\partial \lambda} = 0$

$2x + y - 6 = 0$

$2x + y = 6$ — (3)

Comparing (1) and (2)

$-2y = 2y - 4x$

$-4y = -4x$

$\boxed{y = x}$

putting $x = y$ in (3)

ie $2x + y = 6$

or, $2x + x = 6$

$3x = 6$

$\boxed{x = 2}$

$\therefore \boxed{y = 2}$

S.O.C for maximisation

$$|H| = \begin{vmatrix} \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial^2 L}{\partial x \partial \lambda} \\ \frac{\partial^2 L}{\partial y \partial x} & \frac{\partial^2 L}{\partial y^2} & \frac{\partial^2 L}{\partial y \partial \lambda} \\ \frac{\partial^2 L}{} & \frac{\partial^2 L}{} & \frac{\partial^2 L}{} \end{vmatrix} = \begin{vmatrix} 0 & 4 & 2 \\ 4 & -2 & 1 \\ 2 & 1 & 0 \end{vmatrix}$$

$$\left| \frac{\partial^2 L}{\partial \lambda \partial x} \quad \frac{\partial^2 L}{\partial \lambda \partial y} \quad \frac{\partial^2 L}{\partial \lambda^2} \right| = \left| 2 \quad 1 \quad 0 \right|$$

$$\frac{\partial L}{\partial x} = 4y + 2\lambda$$

$$\frac{\partial^2 L}{\partial x^2} = 0, \quad \frac{\partial^2 L}{\partial x \partial y} = 4$$

$$\frac{\partial^2 L}{\partial x \partial \lambda} = 2$$

$$\frac{\partial L}{\partial y} = 4x - 2y + \lambda$$

$$\frac{\partial^2 L}{\partial y \partial x} = 4 \qquad \frac{\partial^2 L}{\partial y \partial \lambda} = 1$$

$$\frac{\partial^2 L}{\partial y^2} = -2$$

$$\frac{\partial L}{\partial \lambda} = 2x + y - 6$$

$$\frac{\partial^2 L}{\partial \lambda \partial x} = 2 \qquad \frac{\partial^2 L}{\partial \lambda \partial y} = 1 \qquad \frac{\partial^2 L}{\partial \lambda^2} = 0$$

$$|H| = \begin{vmatrix} 0 & 4 & 2 \\ 4 & -2 & 1 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= 0(-2\times 0 - 1\times 1) - 4(4\times 0 - 2\times 1) + 2(4\times 1 - (2\times -2))$$

$$= 0 - 4(-2) + 2(4+4)$$

$$= \quad 1(\ 2) + 2(4+4)$$
$$8 + 16 = 24 > 0$$

S.o.c is satisfied.

$|H| > 0$ (maximised)

at $x = 2$ and $y = 2$ $\quad U = 4xy - y^2$ is maximized

$$U_{max} = 4 \times 2 \times 2 - 2^2$$
$$= 12 \text{ (ans)}$$

Q2 maximise, $U = (x+2)(y+1)$.

Subject to: $4x + 6y - 130 = 0$

Find value of x and y that will maximise U.
What is the maximum value of U.

Q3 maximise $Q = 8x_1^{1/2} + 20x_2^{1/2}$

Subject to, $10 = x_1 + 5x_2$

find x_1, x_2 for max Q.

Q4 max $Q = K^{1/2}L^{1/2}$

subj to $\quad 400 = 8L + 4K$

Find K and L for max Q.