Real Analysis

Sequences and Series.

$S = \{2, 2, \cdots, 2\}$

$S = \{(-1)^n\}$.

Sequences: [Notation $\{x_n\} / \{u_n\}$]

Eg: $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\cdots\}$ $\{\frac{1}{n}\}, n \in \mathbb{N} \Rightarrow \{x_n\} = \{\frac{1}{n}\}$

$S = \{1, 4, 9, \cdots\cdots\}$ $\{n^2\}, n \in \mathbb{N} \Rightarrow \{x_n\} = \{n^2\}$

Types of sequences:

i) Increasing sequence eg: $\{x_n\} = \{n^2\}$.

ii) Decreasing sequence eg: $\{x_n\} = \{\frac{1}{n}\}$.

$\{x_n\} \in \mathbb{R}$.
$n \in \mathbb{N}$

Definitions: Eg: $S = \{1, 4, \underset{n}{9}, \underset{n+1}{16}, 25, \cdots\}$

$x_{n+1} \geq x_n \; \forall n \in \mathbb{N}$

i) A sequence $\{x_n\}$ is said to be increasing if $x_{n+1} \geq x_n \; \forall n \in \mathbb{N}$.
& strictly increasing if $x_{n+1} > x_n \; \forall n \in \mathbb{N}$.

ii) A seq $\{x_n\}$ is said to be decreasing if $x_{n+1} \leq x_n \; \forall n \in \mathbb{N}$
& strictly decreasing if $x_{n+1} < x_n \; \forall n \in \mathbb{N}$

Upper Bound (M) & Lower Bound (m) of a sequences:

i) If seq $\{x_n\}$ has upper bound $\Rightarrow \{x_n\}$ is bounded above.

ii) If seq $\{x_n\}$ has lower bound $\Rightarrow \{x_n\}$ is bounded below.

A seq $\{x_n\}$ is said to be bounded below if $\exists$ a $m \in \mathbb{R}$
s.t $m < x_n \; \forall n \in \mathbb{N}$.

A seq $\{x_n\}$ is said to be bounded above if $\exists$ a $M \in \mathbb{R}$.
s.t $x_n < M \; \forall n \in \mathbb{N}$.

If a seq $\{x_n\}$ has a lower bound & upper bound then it

$\ldots \; x_n \le M \;\forall\, n \in \mathbb{N}$

If a seq $\{x_n\}$ has a lower bound & upper bound, then it is "Bounded sequence".

Eg: $\{x_n\} = \left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\} \Rightarrow$ Bounded

$\{x_n\} = \{n^2\} = \{1, 4, 9, \cdots\} \Rightarrow$ Not bounded.

Q. (i) $\{x_n\} = \left\{\dfrac{2n-1}{3n+4}\right\}$ (ii) $\{x_n\} = \left\{\dfrac{2n+3}{3n+4}\right\}$.

Check for increasing / decreasing seq.

$\forall\, n \in \mathbb{N}$

Inc: $x_{n+1} \ge x_n \Rightarrow x_{n+1} - x_n \ge 0$.

dec: $x_{n+1} \le x_n \Rightarrow x_{n+1} - x_n \le 0$.

(i). $x_n = \dfrac{2n-1}{3n+4}$.

$$x_{n+1} = \frac{2(n+1)-1}{3(n+1)+4} = \frac{2n+1}{3n+7}.$$

$$x_{n+1} - x_n = \frac{2n+1}{3n+7} - \frac{2n-1}{3n+4}$$

$$= \frac{(2n+1)(3n+4) - (2n-1)(3n+7)}{(3n+7)(3n+4)}$$

$$= \frac{\cancel{6n^2} + \cancel{11n} + 4 - (\cancel{6n^2} + \cancel{11n} - 7)}{(3n+7)(3n+4)}$$

$$= \frac{11}{(3n+7)(3n+4)} > 0 \;\Rightarrow \text{strictly increasing.}$$

(ii) $x_n = \dfrac{2n+3}{3n+4}$.

$$x_{n+1} = \frac{2(n+1)+3}{3(n+1)+4} = \frac{2n+5}{3n+7}$$

$$x_{n+1} - x_n = \frac{2n+5}{3n+7} - \frac{2n+3}{3n+4}.$$

$$= \frac{(2n+5)(3n+4) - (2n+3)(3n+7)}{(3n+7)(3n+4)}$$

$$= \frac{(2n+5)(3n+4) - (2n+3)(3n+7)}{(3n+7)(3n+4)}$$

$$= \frac{\cancel{6n^2} + \cancel{2}3n + 20 - (\cancel{6n^2} + \cancel{23n} + 21)}{(3n+7)(3n+4)}$$

$$= \frac{-1}{(3n+7)(3n+4)} < 0 \Rightarrow \text{strictly decreasing.}$$

For decreasing seq, first term is always the largest term of the seq.

$$x_n = \frac{2n+3}{3n+4} \Rightarrow x_1 = \frac{5}{7} \Rightarrow x_n \leq x_1 = \frac{5}{7} \ \forall n \in \mathbb{N}$$

$$x_n = \frac{2n+3}{3n+4} = \frac{\frac{2}{3}(3n+4-4) + 3}{3n+4}$$

$$= \frac{\frac{2}{3}(3n+4) - \frac{8}{3} + 3}{3n+4}$$

$$= \left(\frac{2}{3}\right) + \left(\frac{1/3}{3n+4}\right) > \frac{2}{3}$$

$$x_n > \frac{2}{3} \ \forall n \in \mathbb{N}$$

$$\therefore \ \frac{2}{3} < x_n \leq \frac{5}{7} \ \forall n \in \mathbb{N} \Rightarrow \text{Bounded sequence.}$$

Q. $\{x_n\} = \left\{ (-1)^n \, \frac{3n-1}{n} \right\}$ ...... [Alternating seq].

Check for Increasing / Decreasing seq & check for boundedness.

$$x_n = (-1)^n \frac{3n-1}{n}$$

$$x_{n+1} = (-1)^{n+1} \frac{3(n+1)-1}{n+1} = (-1)^{n+1} \cdot \frac{3n+2}{n+1}$$

$$x_{n+1} - x_n = (-1)^{n+1} \frac{3n+2}{n+1} - (-1)^n \frac{3n-1}{n}$$

$$= (-1)^n(-1)\left(\frac{3n+2}{n+1}\right) - (-1)^n \frac{3n-1}{n}.$$

$$= -(-1)^n\left[\frac{3n+2}{n+1} + \frac{3n-1}{n}\right].$$

$$= (-1)^{n+1}\left[\frac{3n^2+2n+(3n-1)(n+1)}{n(n+1)}\right].$$

$$= (-1)^{n+1}\left[\frac{3n^2+2n+3n^2+2n-1}{n(n+1)}\right].$$

$$= (-1)^{n+1}\left[\frac{6n^2+4n-1}{n(n+1)}\right] \gtrless 0. (\times)$$

$$x_n = (-1)^n \frac{3n-1}{n}.$$

$$|x_n| = \left|(-1)^n \cdot \frac{3n-1}{n}\right|.$$

$$= |(-1)^n| \left|\frac{3n-1}{n}\right|. = \left|\frac{3n-1}{n}\right|.$$

$$|x_n| = \left|3 - \frac{1}{n}\right|$$

$$= 3 - \left(\frac{1}{n\uparrow}\right)_{\downarrow} < 3.$$

$$\Rightarrow \quad |x_n| < 3 \quad \Rightarrow \quad -3 < x_n < 3 \quad \forall\, n \in \mathbb{N}.$$

$\{x_n\}$: Bounded sequence.

$$x_n = (-1)^n \frac{3n-9}{n} \quad \Rightarrow \quad 3(-1)^n \frac{n-3}{n}.$$