Problem: Calculate

i) Probability
ii) Mean
iii) Variance
iv) Mgf

Soln:

Develop a mathematical construct of the problem

Approximate the mathematical construct with any of the following:

1) Standard Normal (Z-distribution: Table)
2) Chi-square (Table)
3) t- Statistics (Tables)

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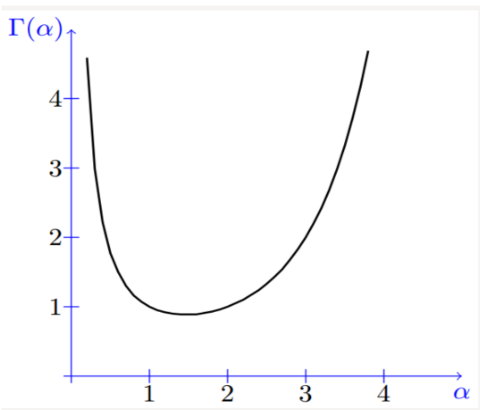
Gamma Distribution: *Definition and properties, probability distribution, mean, variance, moment generating function, cumulant generating function, additive property and limiting case. Examples and applications based on the distribution.*

Gamma Function

If for any positive real number (∝), a function f(x) is said to follow gamma function, if:

$$\Gamma(\propto) = \int_0^{\infty} x^{\propto-1} e^{-x} dx, \quad for\ \propto > 0$$

The following figure shows the gamma function for a positive real number (∝):



Properties of gamma function:

For any positive real number α:

1. $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx;$
2. $\int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}, \qquad \text{for } \lambda > 0;$
3. $\Gamma(\alpha+1) = \alpha\Gamma(\alpha);$
4. $\Gamma(n) = (n-1)!, \text{ for } n = 1, 2, 3, \cdots;$
5. $\Gamma(\frac{1}{2}) = \sqrt{\pi}.$

The most important property is however,

Γ(n) = (n-1)!, for n= 1,2,3,..........................

Gamma Distribution

$A\ continuous\ r.v\ \ X\ is\ said\ to\ have\ a\ gamma\ distribution\ with\ parameters\ \alpha > 0\ and\ \lambda > 0,\ \ shown\ as\ X \backsim \Gamma(\alpha,\lambda), if\ its\ pdf\ is\ given\ by:$

$$f(x) = \frac{\lambda^{\propto} x^{\propto-1} e^{-\lambda x}}{\Gamma(\alpha)}; when\ x > 0$$

$$= 0, \qquad Otherwise$$

If we let, $\propto = 1, we\ obtain:$

$$f(x) = \ \lambda e^{-\lambda x}, when\ x > 0$$

$$= 0,\ \ Otherwise$$

Show that $\Gamma(1, \lambda) = \lambda!$

Ans: we know

$$f(x) = \frac{\lambda^{\propto} x^{\propto - 1} e^{-\lambda x}}{\Gamma(\alpha)}; when\ x > 0$$

$$= 0, \qquad Otherwise$$

Now, we have

$$Parameters\ values: \alpha = 1, \lambda$$

So,

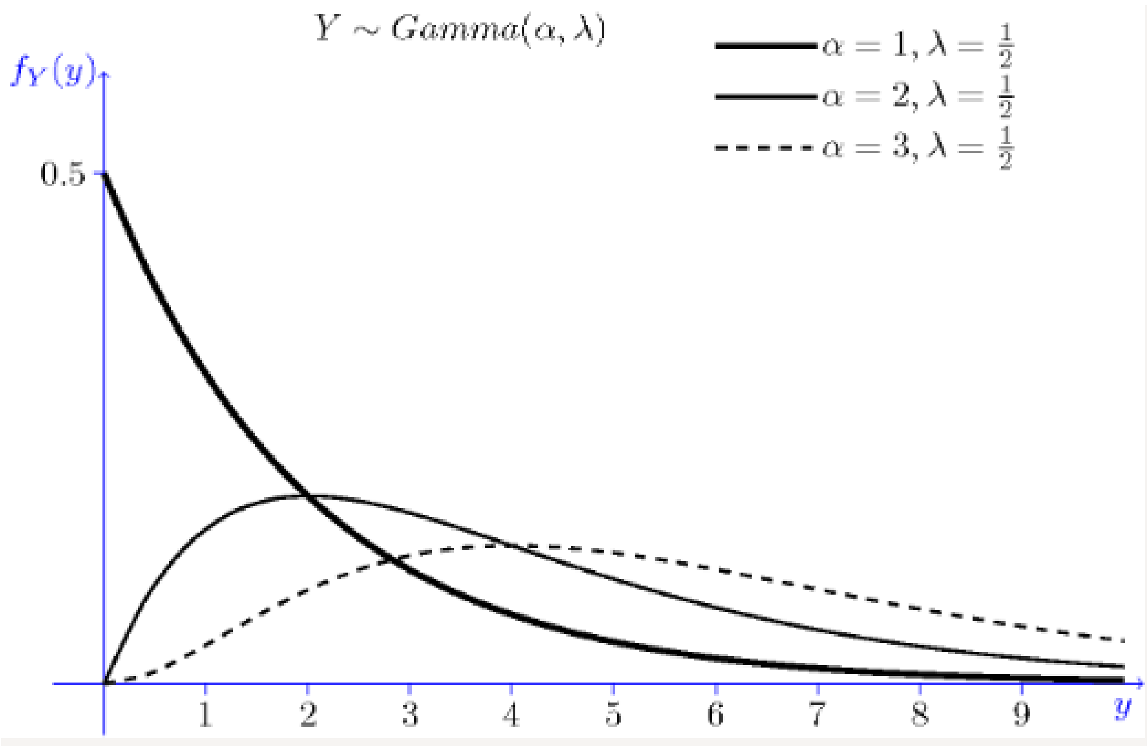
$$f(x) = \frac{\lambda \ \ e^{-\lambda x}}{\Gamma(1)}; when\ x > 0$$

$$= 0, \qquad Otherwise$$

$$f(x) = \lambda e^{-\lambda x}, when\ x > 0$$

$$= \Gamma(\lambda + 1) = \lambda!$$

The "Probability Curve" of gamma distribution:



Problem 1:

Using the properties of the gamma function, show that the gamma PDF integrates to 1, i.e., show that for $\alpha, \lambda > 0$, we have

$$\int_0^\infty \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx = 1.$$

Ans.

$$\int_0^\infty \frac{\lambda^\propto x^{\propto-1} e^{-\lambda x}}{\Gamma(\alpha)} \, dx$$

$$= \lambda^\propto \int_0^\infty \frac{x^{\propto-1} e^{-\lambda x}}{\Gamma(\alpha)} \, dx = \lambda^\propto . \left\{ \frac{\Gamma(\alpha)}{\lambda^\alpha \Gamma(\alpha)} \right\} = 1$$

We know,

$$2. \int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}, \qquad \text{for } \lambda > 0;$$

Moment Generating Function (mgf) of Gamma Distribution

We know, for a pdf (f(x)), the r-th 'Moment Generating Function (mgf) is defined as:

$$m_r(x) = E(e^{rx})$$

$$for\ all\ 'r' such\ that\ this\ expectation\ exists.$$

In case of gamma distribution:

$$Let\ X \sim \Gamma(\propto, \beta) for\ some\ \alpha, \beta > 0; the\ mgf\ of\ X\ is\ given\ by:$$

$$\boldsymbol{m_r(x)} = \left(1 - \frac{r}{\beta}\right)^{-\alpha}, when\ r < \beta$$

$$: does\ not\ exist\ , when\ r \geq \beta$$

$$Let\ X \sim \Gamma(\propto, \beta) for\ some\ \alpha, \beta > 0; the\ 'Mean' and\ 'Variance' of\ X\ is\ giveb\ by:$$

$\boldsymbol{Mean}$: $E[x] = \propto \beta$

$\boldsymbol{Variance}$: $V(x) = E(x^2) - \{E(x)\}^2 = \alpha\beta^2$