Math eco chapter1: derivatives, maxima and minima of single variable functions with economic applications.)

23 May 2023

Maxima and Minima (single variable function)

$$\boxed{y = f(x)}$$

x?

	Necessary (F.O.C)	Sufficiency (S.O.C)
① Maxima →	$dy/dx = 0$ or $f'(x) = 0$	$d^2y/dx^2 < 0$ or $f''(x) < 0$
② Minima	$dy/dx = 0$ or $f'(x) = 0$	$d^2y/dx^2 > 0$ or $f''(x) > 0$

③ Point of Inflexion (Change in curvature) (Second order)

- Stationary: $f'(x) = 0$; $f''(x) = 0$ and $f'''(x) \neq 0$
- Non-Stationary: $f'(x) \neq 0$; $f''(x) = 0$ & $f'''(x) \neq 0$

Economic Application (Cost and Profit).

$C = f(Q)$

cost ↑ output ↑

$$\boxed{TC = TFC + TVC}$$

$$AC = \frac{TC}{Q} = \frac{TFC}{Q} + \frac{TVC}{Q}$$

$$\boxed{AC = AFC + AVC}$$

and $\boxed{MC = \frac{dTC}{dQ}}$

Note:

- Minimisation of cost
- for what value of Q
- F.O.C & S.O.C

Profit:

$\Pi = TR - TC$

$\boxed{\Pi \quad P.Q - C(Q)}$

Maximisation of Π for what value of q.

Questions:

Firm ⇐ cost minimum

1. **Illustration 3.**
Obtain the extrema for the function $C = q^3 + 2q^2 - 4q + 4$

maximum or minimum; $C = f(q)$; cost, output; q = ? C max or min (minimisation)

2. **Illustration 5.4**
Obtain the inflexion point for the function $y = x^3 + 6x^2 + 5x - 30$. Examine whether this point is a stationary or a non-stationary one.

$f'(x) = 0$; $f'(x) \neq 0$; $x = -2$

3. **Example 93.**
Total cost is given by $C = 5000 + 1000q - 500q^2 + \frac{2}{3}q^3$ (i) Find the MC function. (ii) Find the expression for the slope of the MC curve. (iii) Find the average total cost function. (iv) At what value of q does MC equal AVC ? [D. U. 1992]

FC ; TVC ; AC, AVC ; q^* = ? at AVC = MC

4. **Example 89.**
A competitive firm sells its output at a fixed price of Rs. 4 per unit. The cost function of the firm is given as : $C = 0.04q^3 - 0.9q^2 + 10q + 5$. Find the profit-maximising output level of the firm and determine the corresponding total profit, total revenue and total variable cost. Do you think the firm will continue production ?

$q = 10$; $P = 4$; q ; $C = 1$
$\Pi = TR - C$; $TR = 4 \cdot q$

5. **Example 104.**
The total cost function of a firm is $C = 2q^3 - 3q^2 + 12q$. Show that at the minimum point of the AC curve the average cost equals the marginal cost.

HW.

Example 88.

Given the total cost function $C = 15q - 6q^2 + q^3$ derive the equation of AC and MC curves. Find the output level at which AC is minimum and show that when AC is minimum AC = MC.

Solutions:

(1) $C = q^3 + 2q^2 - 4q + 4$

F.O.C $\quad \frac{dc}{dq} = 0 \Rightarrow 3q^2 + 4q - 4 = 0$

$\Rightarrow 3q^2 + 6q - 2q - 4 = 0$

$\Rightarrow 3q(q+2) - 2(q+2) = 0$

$\Rightarrow (3q-2)(q+2) = 0$

$\therefore q = 2/3$ or $q = -2$

S.O.C $\quad \frac{d^2c}{dq^2} = 6q + 4$

at $q = 2/3 \Rightarrow \frac{d^2c}{dq^2} = \cancel{6} \times \frac{2}{\cancel{3}} + 4 = 8 > 0 \Rightarrow$ at $q = 2/3$ c is minimised.

and at $q = -2 \Rightarrow \frac{d^2c}{dq^2} = 6(-2) + 4 = -12 + 4 = -8 \Rightarrow$ at $\boxed{q = -2}$ c is maximised.

(2) $y = x^3 + 6x^2 + 5x - 30$

To find the point of inflexion: (i) $f'(x)$
(ii) $\boxed{f''(x) = 0}$
(iii) $f'''(x) \neq 0$

$\Rightarrow \frac{dy}{dx} = 3x^2 + 12x + 5$

At point of inflexion $\frac{d^2y}{dx^2} = 0$

$\Rightarrow 6x + 12 = 0$

$\Rightarrow \boxed{x = -2}$

inflexion is

$\Rightarrow \boxed{x = -2}$ ✓

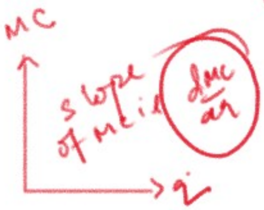
Now, $\frac{d^3y}{dx^3} = 6 \neq 0$ $\quad\therefore$ The point of inflexion is at $x = -2$.

→ at $x = -2$, $\frac{dy}{dx} = 3(-2)^2 + 12(-2) + 5$
$= 12 - 24 + 5 = 17 - 24 = -7 \neq 0$ (Non stationary).

— * —

(3) Given $C = \underbrace{(5000)}^{TFC} + \overbrace{\left(1000q - 500q^2 + \frac{2}{3}q^3\right)}^{TVC}$

(i) To find the equation of MC curve,



$MC = \frac{dC}{dq} = 1000 - 1000q + \frac{2}{3} \times 3q^2$

$\boxed{MC = 1000 - 1000q + 2q^2}$ ✓

(ii) To find the slope of MC,

$\boxed{\frac{dMC}{dq} = -1000 + 4q}$

(iii) To find avg total cost (AC).

given, $C = 5000 + \left(1000q - 500q^2 + \frac{2}{3}q^3\right)$

$\boxed{AC = \frac{C}{q} = \frac{5000}{q} + \left[1000 - 500q + \frac{2}{3}q^2\right]}$ ✓

(iv) $q = ?$ when AVC = MC

$AVC = \frac{TVC}{q} = \frac{1000q - 500q^2 + 2/3\, q^3}{q}$

$1\Rightarrow AVC = 1000 - 500q + \frac{2}{3}q^2$

Acc to the question: $AVC = MC$

$$\cancel{1000} - 500q + \frac{2}{3}q^2 = \cancel{1000} - 1000q + 2q^2$$

$$1000q - 500q = 2q^2 - \frac{2}{3}q^2$$

$$500q = \frac{4q^2}{3}$$

$$3 \times 500\cancel{q} = 4q^{\cancel{2}}$$

$$\frac{3 \times 500}{4} = q$$

$$\boxed{q = 375 \text{ units}}$$

(4) Acc to ques $TR = P \cdot q = 4q$ — (1).

and $C = 0.04q^3 - 0.9q^2 + 10q + 5$ — (2)

$\therefore$ Profit fn of the firm is

$$\pi = TR - TC$$

$$\pi = 4q - (0.04q^3 - 0.9q^2 + 10q + 5)$$

$$\pi = 4q - 0.04q^3 + 0.9q^2 - 10q - 5$$

$$\boxed{\pi = -0.04q^3 + 0.9q^2 - 6q - 5}$$

To maximise π

F.O.C $\quad \frac{d\pi}{dq} = 0$

F.O.C $\frac{d\pi}{dq} = 0$

$-0.24q+1.8$

$\Rightarrow (-0.12q^2 + 1.8q - 6) = (0) \times 100$

$\Rightarrow -12q^2 + 180q - 600 = 0$

$\Rightarrow -q^2 + 15q - 50 = 0$

$\Rightarrow q^2 - 15q + 50 = 0$

$\Rightarrow q^2 - 10q - 5q + 50 = 0$

$\Rightarrow q(q-10) - 5(q-10) = 0$

$\Rightarrow (q-5)(q-10) = 0$

$\therefore q = 5$ or $q = 10$

S.O.C: $\frac{d^2\pi}{dq^2} = -0.24q + 1.8$

at $q = 5 \Rightarrow \frac{d^2\pi}{dq^2} = -0.24(5) + 1.8$
$= 0.6 > 0$

and at $q = 10 \Rightarrow \frac{d^2\pi}{dq^2} = -0.24(10) + 1.8$
$= -0.6 < 0$ (this satisfies S.O.C for max)

(ans) $\therefore$ at $q = 10$ profit is maximised