Q. Draw the isoquant and obtain the cost function

$q = AL^{\alpha}K^{\beta}$, $A, \alpha, \beta > 0$ [Parameters] . Plot the cost fn.

Isoquants: -vely sloped and convex to the origin [obtained!]

Cost function: $\boxed{\min_{\{L,K\}} wL + rK \quad s.t \quad \bar{q} = q(L,K)}$

Setup the Lagrangian:

$$\mathcal{L} = wL + rK + \lambda[\bar{q} - AL^{\alpha}K^{\beta}], \quad \lambda = \text{Lagrange Multiplier.}$$

FOC: $\frac{\partial \mathcal{L}}{\partial L} = 0 \Rightarrow w - \lambda A \alpha L^{\alpha-1}K^{\beta} = 0 \Rightarrow w = \lambda A \alpha L^{\alpha-1}K^{\beta}$ ... (i)

$\frac{\partial \mathcal{L}}{\partial K} = 0 \Rightarrow r - \lambda A \beta L^{\alpha}K^{\beta-1} = 0 \Rightarrow r = \lambda A \beta L^{\alpha}K^{\beta-1}$ ... (ii)

$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow \bar{q} = AL^{\alpha}K^{\beta}$ ... (iii)

(i) : $\frac{w}{A\alpha L^{\alpha-1}K^{\beta}} = \lambda$ (ii) $\frac{r}{A\beta L^{\alpha}K^{\beta-1}} = \lambda$

combining: $\frac{w}{\cancel{A}\alpha L^{\alpha-1}K^{\beta}} = \frac{r}{\cancel{A}\beta L^{\alpha}K^{\beta-1}} \Rightarrow \frac{w}{r} = \frac{\alpha L^{\alpha-1}K^{\beta}}{\beta \cdot L^{\alpha}K^{\beta-1}}$

$\Rightarrow \boxed{\frac{w}{r} = \left(\frac{\alpha}{\beta}\right)\left(\frac{K}{L}\right)}$ .... [Note: $\frac{MP_L}{MP_K} = \frac{w}{r}$] .

$\boxed{K = \left(\frac{w}{r}\right)\left(\frac{\beta}{\alpha}\right) \cdot L}$ ..... (iv)

Putting in (iii) :- $q = AL^{\alpha}K^{\beta}$.

$$q = AL^{\alpha}\left\{\left(\frac{w}{r}\right)\left(\frac{\beta}{\alpha}\right) \cdot L\right\}^{\beta}.$$

$$q = A\left(\frac{w}{r}\right)^{\beta}\left(\frac{\beta}{\alpha}\right)^{\beta}L^{\alpha+\beta}.$$

$$\left(\frac{q}{A}\right)\left(\frac{r}{w}\right)^{\beta}\left(\frac{\alpha}{\beta}\right)^{\beta} = L^{\alpha+\beta}.$$

$$L^* = \left(\frac{q}{A}\right)^{\frac{1}{\alpha+\beta}} \left\{\left(\frac{r}{w}\right)\left(\frac{\alpha}{\beta}\right)\right\}^{\frac{\beta}{\alpha+\beta}}.$$

Put L^* in (iv):-

$$K^* = \left(\frac{w}{r}\right)\left(\frac{\beta}{\alpha}\right)\cdot L^*.$$

$$K^* = \left(\frac{w}{r}\right)\left(\frac{\beta}{\alpha}\right)\cdot\left(\frac{q}{A}\right)^{\frac{1}{\alpha+\beta}} \left\{\left(\frac{r}{w}\right)\left(\frac{\alpha}{\beta}\right)\right\}^{\frac{\beta}{\alpha+\beta}}.$$

$$K^* = \left(\frac{q}{A}\right)^{\frac{1}{\alpha+\beta}} \underbrace{\left(\frac{w}{r}\right)\cdot\left(\frac{w}{r}\right)^{-\frac{\beta}{\alpha+\beta}}} \cdot \left(\frac{\beta}{\alpha}\right)\left(\frac{\beta}{\alpha}\right)^{-\frac{\beta}{\alpha+\beta}}.$$

$$K^* = \left(\frac{q}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{w}{r}\right)^{1-\frac{\beta}{\alpha+\beta}} \left(\frac{\beta}{\alpha}\right)^{1-\frac{\beta}{\alpha+\beta}}.$$

$$1-\frac{\beta}{\alpha+\beta} = \frac{\alpha+\cancel{\beta}-\cancel{\beta}}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta}$$

$$K^* = \left(\frac{q}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{w}{r}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}.$$

Now $C = wL^* + rK^*$

$$= w\cdot\left(\frac{q}{A}\right)^{\frac{1}{\alpha+\beta}} \left\{\left(\frac{r}{w}\right)\left(\frac{\alpha}{\beta}\right)\right\}^{\frac{\beta}{\alpha+\beta}} + r\cdot\left(\frac{q}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{w}{r}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}$$

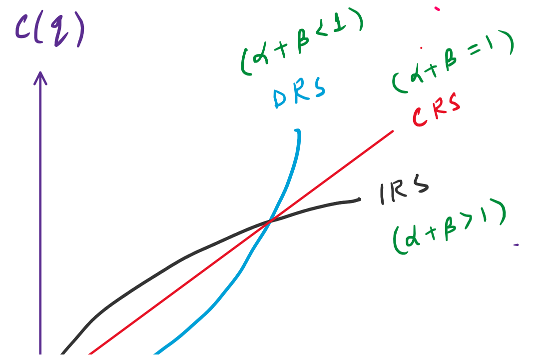
$$= \left(\frac{q}{A}\right)^{\frac{1}{\alpha+\beta}} \cdot \underbrace{\left[w\cdot\left\{\left(\frac{r}{w}\right)\left(\frac{\alpha}{\beta}\right)\right\}^{\frac{\beta}{\alpha+\beta}} + r\left\{\left(\frac{w}{r}\right)\left(\frac{\beta}{\alpha}\right)\right\}^{\frac{\alpha}{\alpha+\beta}}\right]}_{\text{Exp of } w,\, r,\, \alpha,\, \beta \text{ -- Parameters.}}$$

$$= \left(\frac{q}{A}\right)^{\frac{1}{\alpha+\beta}}\cdot k, \quad \text{where } k = [\cdots].$$

$$= q^{\frac{1}{\alpha+\beta}}\cdot A^{-(\alpha+\beta)}\, k.$$

$$= k_1\, q^{1/\alpha+\beta}, \quad k_1 = A^{-(\alpha+\beta)}\cdot k.$$

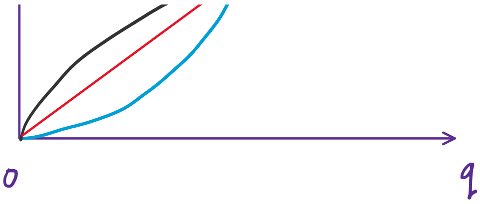
Cost fn: $\boxed{C = C(q) = k\cdot q^{\frac{1}{\alpha+\beta}}}$



$$C = k\cdot q^{1/\alpha+\beta}, \quad \alpha, \beta > 0$$

$$MC = \frac{dC}{dq} = k\cdot\frac{1}{\alpha+\beta}\cdot q^{\left(\frac{1}{\alpha+\beta}-1\right)} > 0.$$

$$MC = \frac{dC}{dq} = k \cdot \underbrace{\frac{1}{\alpha+\beta}} \cdot q \quad > 0 .$$



$$\frac{d^2C}{dq^2} = \underbrace{k}_{>0} \cdot \underbrace{\frac{1}{\alpha+\beta}}_{>0} \underbrace{\left(\frac{1}{\alpha+\beta} - 1\right)}_{\left\{\gtreqless 0\right\}} \underbrace{q^{\left(\frac{1}{\alpha+\beta}-2\right)}}_{>0} .$$

(i) $\frac{1}{\alpha+\beta} - 1 > 0 \Rightarrow \frac{1}{\alpha+\beta} > 1 \Rightarrow \boxed{\alpha+\beta < 1}$ -- DRS, $\frac{d^2C}{dq^2} > 0$.

(convex) .

(ii) $\frac{1}{\alpha+\beta} - 1 < 0 \Rightarrow \frac{1}{\alpha+\beta} < 1 \Rightarrow \boxed{\alpha+\beta > 1}$ .. IRS, $\frac{d^2C}{dq^2} < 0$

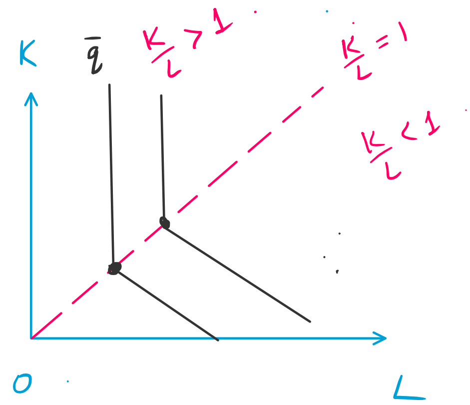
(concave)

(iii) $\frac{1}{\alpha+\beta} - 1 = 0 \Rightarrow \frac{1}{\alpha+\beta} = 1 \Rightarrow \boxed{\alpha+\beta = 1}$ -- CRS . $\frac{d^2C}{dq^2} = 0$.

(st. line) .

Q. Find the isoquant map & cost fn given:

$$q = \min\{2L, L+K\} .$$



At opt : $2L = L + K$.

$L = K \Rightarrow \frac{K}{L} = 1$

Case I : $2L < L + K \Rightarrow L < K \Rightarrow \left(\frac{K}{L} > 1\right)$

$q = 2L$.

Fix $q = \bar{q} \Rightarrow \bar{q} = 2L$

$d\bar{q} = 2 \cdot dL \Rightarrow dL = 0$.

Slope of isoquant: $\frac{dK}{dL} \to \infty$.

Case II : $L + K < 2L \Rightarrow K < L \Rightarrow L > K \Rightarrow \left(\frac{K}{L} < 1\right)$

$q = L + K$.

Fix $q = \bar{q} \Rightarrow \bar{q} = L + K$.

Diff: $d\bar{q} = dL + dK$.

Slope of isoquant: $\frac{dK}{dL} = -1$

Cost fn: At opt: $2L = L + K = q$

$\Rightarrow L = K = q$

$C = wL + rK = w \cdot q + r \cdot q = (w + r) \cdot q$.

(Cost fn)

HW: Q. Draw the isoquant map and obtain the cost fn for

$q = \min\{L + K, 2K\}$.

Q. Consider a production setup, with 4 factors of production: x_1, x_2, x_3, x_4 with their mkt wage rates: w_1, w_2, w_3, w_4. The prodn fn is given by $q = \min\{x_1, x_2\} + \min\{x_3, x_4\}$. Find the cost fn.

Cost Exp $C = w_1 x_1 + w_2 x_2 + w_3 x_3 + w_4 x_4$

On replacing with $x_1^*, x_2^*, x_3^*, x_4^*$ we will get the cost fn.

$$q = \underbrace{\min\{x_1, x_2\}}_{\text{P.C b/w } x_1, x_2} + \underbrace{\min\{x_3, x_4\}}_{\text{P.C b/w } x_3, x_4}$$

P.S relation $(x_1, x_2) / (x_3, x_4)$

Let $\min\{x_1, x_2\} = a$, $w_a = w_1 + w_2$ [At opt: $x_1^* = x_2^*$]

Let $\min\{x_3, x_4\} = b$, $w_b = w_3 + w_4$ [At opt: $x_3^* = x_4^*$]

$q = a + b$ ..... [P.S prodn fn].

[Producer will employ the factor which is cheaper].

Case I: If $w_a < w_b \Rightarrow b^* = 0$, $a^* = q$

[producer will employ the factor which is cheaper] .

Case I: If $w_a < w_b \Rightarrow b^* = 0$., $a^* = q$.

$$\begin{array}{cc} \downarrow & \downarrow \\ x_3^* = x_4^* = 0 . & x_1^* = x_2^* = q . \end{array}$$

$$\text{Cost fn} = w_1 x_1^* + w_2 x_2^* = \boxed{(w_1 + w_2)\, q = c_1}$$

Case II: If $w_b < w_a \Rightarrow a^* = 0$, $b^* = q$

$$\begin{array}{cc} \downarrow & \downarrow \\ x_1^* = x_2^* = 0 & x_3^* = x_4^* = q \end{array}$$

$$\text{Cost fn} = w_3 x_3^* + w_4 x_4^* = \boxed{(w_3 + w_4) \cdot q = c_2}$$

$$\text{Cost fn: } C = \min\{c_1, c_2\}$$

$$C = \min\{(w_1 + w_2)\, q,\ (w_3 + w_4) \cdot q\} .$$

$$\boxed{C = q \min\{(w_1 + w_2), (w_3 + w_4)\}}$$