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Sequence of Real Numbers

① Natural number $\{1, 2, \dots\}$

② Integers $\{\dots -2, -1, 0, 1, 2, \dots\}$

③ Bounds of a Set

given a $\boxed{\text{set } S}$ of real no.s, if

(a) there exist a number G such that $x \le G$, for every number x of S. Here G is the upper bound of set S.

(b) $y \ge G \Rightarrow$ G is the lower bound.

Sequence of numbers.

$$\boxed{f : N \to R}$$

a) finite sequence : $\{3, 5, 7, 9, 11\}$

(b) infinite sequence :

(i) $\{n^2\} = \{1^2, 2^2, 3^2, \dots, n^2, \dots\}$

(ii) $\left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, \dots\right\}$

$\ldots \}$

Ex: Bounded sequence:

1. The sequence $\{n+1\} = \{2, 3, 4, \ldots\}$

it is bounded below.

maximum

2. The sequence $\left\{1+\frac{1}{n}\right\} = \left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \ldots\right\}$

$1+\frac{1}{1} = 1+1 = 2$ $\quad$ $1+\frac{1}{2} =$

$<2 \quad < 3/2 \quad < 5/4 \quad 4/3 \ldots$

bounded above.

increasing $>$
non decreasing $\geq$

3. Monotonic Sequence:

(a) A sequence $\{x_n\}$ is said to be monotonic increasing or monotonic non-decreasing

$x_n \leq x_{n+1}$

$\{x_1 \leq x_2 \leq x_3 \leq \ldots\}$ for all $n \in N$

A sequence $\{x_n\}$ is said to be monotonically strictly increasing if

$x_n < x_{n+1}$ for all $n \in N$.

4. Monotonically Decreasing function.

4. Monotonically Decreasing function.

A sequence $\{x_n\}$ if $x_n \geq x_{n+1}$ for all $n \in N$

$$\{x_1 \geq x_2 \geq x_3 \geq x_4 \geq \cdots\cdots\}$$

monotonically strictly decreasing function.

$x_n > x_{n+1}$ for all $n \in N$.

$$\{x_1 > x_2 > x_3 > \cdots\cdots\}$$

1) $\{n\} = \{1, 2, 3, \cdots\cdots\cdots\}$ monotonically increasing

2) $\{2^n\} = \{2, 4, 8, 16, \cdots\cdots\}$ "

3) $\{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\cdots\}$

4) $\{\frac{1}{2^n}\} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\cdots\}$

Monotonically decreasing sequence!

(5) $\{(-1)^{n+1}\} = \{1, -1, 1, -1, \cdots\cdots\}$

is neither monotonically increasing nor decreasing

↳ is neither increasing nor decreasing sequence.

5. Limit of a Sequence:

An infinite sequence $\{x_n\}$ is said to have a finite limit l, if for any pre assigned positive number ϵ, however small, there corresponds a positive integer N, such that $|x_n - l| < \epsilon$ for $n > N$.

This is expressed as

$$\lim_{n\to\infty} x_n = l$$

Find the limit of the sequence $\left\{\frac{1}{n}\right\}$ as $n\to\infty$

we observe that here $\left|\frac{1}{n} - 0\right| < \varepsilon$

for $n < \frac{1}{\varepsilon}$ N

Taking $N = \frac{1}{\varepsilon}$, we can write

$$\left|\frac{1}{n} - 0\right| < \varepsilon \text{ if } n \geq N$$

$\therefore \lim_{n\to\infty} \frac{1}{n} = 0.$

Show that $\lim_{n\to\infty} x_n = 1$, where $x_n = 1 + \frac{(-1)^2}{n}$

If $\varepsilon > 0$, $|x_n - 1| = \left|1 + \frac{(-1)^n}{n} - 1\right| = \left|\frac{(-1)^n}{n}\right|$

$= \frac{1}{n} < \varepsilon$ if $n > \frac{1}{\varepsilon}$

any positive integer $M > \frac{1}{\varepsilon}$, then $|x_n - 1| < \varepsilon$ for all $n \geq M$

$\boxed{\lim_{n\to\infty} x_n = 1}$ ℓ

↳ convergent sequence

(*) Convergent Sequence :

An infinite sequence $\{x_n\}$ is said to convergent if the sequence has a finite limit 'ℓ' i.e. if corresponding to any arbitrary

'l' ie if corresponding to any arbitrary small positive number ε, we can find a positive integer N, depending on ε, such that

$$|x_n - l| < \varepsilon \text{ for } n \geq N$$

ie, $l - \varepsilon < x_n < l + \varepsilon$, when $n \geq N$.

We say $\{x_n\}$ converges to this limit l.
It is written as $\lim_{n \to \infty} x_n = l$

Ex: The sequence $\{x_n\}$, where $x_n = 2 - \frac{1}{2^n}$ is convergent.

Here, $|x_n - 2| = \left| 2 - \frac{1}{2^n} - 2 \right| = \left| -\frac{1}{2^n} \right|$

$$= \frac{1}{2^n}$$

Let $\varepsilon > 0$

Then $|x_n - 2| < \varepsilon$ if $\frac{1}{2^n} < \varepsilon$

or $2^n > \frac{1}{\varepsilon}$

$$\text{or } \boxed{2^n > \frac{1}{\varepsilon}}$$

Taking log on both sides, $n \log 2 > \log\left(\frac{1}{\varepsilon}\right)$

$$n > \frac{\log 1/\varepsilon}{\log 2}$$

If M any positive integer such that $M \geq \frac{\log 1/\varepsilon}{\log 2}$

$$\boxed{\text{then } |x_n - 2| < \varepsilon, \text{ for } n \geq M}$$

$$\therefore \lim_{n \to \infty} x_n = 2$$

$\therefore$ Thus the sequence $\{x_n\}$ is convergent and it converges to 2.

Non-convergent Sequence:

A sequence $\{x_n\}$ is said to diverge to $+\infty$ if for any number G, however large, is assigned, there corresponds a positive integer N, such that $x_n > G$, for all $n \geq N$

integer N, such that $x_n > g$, for all $n > N$.

ie symbolically as $\lim_{n \to \infty} x_n = \infty$

then the sequence is called divergent.

A sequence $\{x_n\}$ is said to diverge to $-\infty$, if when any no. g whatever is assigned, there always exists a positive integer N, such that $x_n < g$ for all $n \geq N$

generally a -ve no.

symbolically, $\lim_{n \to \infty} x_n = -\infty$

Divergent Sequence are sequence which diverges either to $-\infty$ or $+\infty$.

ex: $\{2^n\} = \{2, 4, 8, 16, \ldots\} \to \infty$

$\{-n^2\} = \{-1, -4, -9, -16, \ldots\} \to -\infty$