General soln

Particular soln

Q. Solve: $y - x \cdot \frac{dy}{dx} = k\left(y^2 + x^2 \frac{dy}{dx}\right)$, where $y = k$, $x = k$.

$$y - ky^2 = k \cdot x^2 \frac{dy}{dx} + x \cdot \frac{dy}{dx}$$

$$(y - ky^2) = (kx^2 + x) \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dx}{kx^2 + x} = \frac{dy}{y - ky^2} \quad \ldots\ldots \text{[variable separation]}$$

Integrate: $$\int \frac{dx}{kx^2 + x} = \int \frac{dy}{y - ky^2} \quad \text{------} \quad (*)$$

$$\int \frac{1/k}{x^2 + \frac{x}{k}}\, dx = -\int \frac{1/k}{y^2 - y/k}\, dy$$

$$\int \frac{1}{x^2 + \frac{x}{k} + \frac{1}{4k^2} - \frac{1}{4k^2}}\, dx = -\int \frac{1}{y^2 - \frac{y}{k} + \frac{1}{4k^2} - \frac{1}{4k^2}}\, dy$$

$$\int \frac{1}{\left(x + \frac{1}{2k}\right)^2 - \left(\frac{1}{2k}\right)^2}\, dx = \int \frac{1}{\left(y - \frac{1}{2k}\right)^2 - \left(\frac{1}{2k}\right)^2}\, dy$$

From $(*)$: $$\int \frac{dx}{x(1 + kx)} = \int \frac{dy}{y(1 - ky)}$$

Partial Fractions.

$$\frac{1}{x(1+kx)} = \frac{A}{x} + \frac{B}{1+kx} \Rightarrow \quad 1 = A(1 + kx) + Bx$$

$$1 = A + x(Ak + B)$$

compare coeff: $A = 1$.

$$Ak + B = 0$$

$$k + B = 0 \Rightarrow B = -k$$

$$\frac{1}{x(1+kx)} = \frac{1}{x} - \frac{k}{1+kx}$$

Similarly $\dfrac{1}{y(1-ky)} = \dfrac{1}{y} + \dfrac{k}{1-ky}$.

$$\int \frac{1}{x}\,dx - k\int \frac{1}{1+kx}\,dx = \int \frac{1}{y}\,dy + k\int \frac{1}{1-ky}\,dy$$

$$\ln x - \ln|1+kx| = \ln|y| - \ln|1-ky| + c.$$

$$\ln\left|\frac{x}{1+kx}\right| = \ln\left|\frac{y}{1-ky}\right| + c \quad \text{.... (General soln)}$$

$$\ln\left|\frac{x(1-ky)}{y(1+kx)}\right| = c$$

$$\frac{x(1-ky)}{y(1+kx)} = e^{c} = A.$$

Given $x = k,\ y = k,\quad A = \dfrac{1-k^2}{1+k^2}$

$\therefore$ Soln is : $\dfrac{x(1-ky)}{y(1+kx)} = \dfrac{1-k^2}{1+k^2}$ ... [Particular soln]

(III) Homogeneous differential Equations:-

Homogeneous fn:

Eg: $u = f(x, y)$

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

$\Rightarrow f(x, y)$ is homogeneous of degree n.

Suppose $\dfrac{dy}{dx} = f(x, y)$: $f(x, y)$ is a homogeneous fn.

Q. Solve: $2xy\,dy = (x^2 + 3y^2)\,dx$ ↪ variable (x, y)

[variable sep not directly possible]

Q. Solve: $2xy\, dy = (x^2 + 3y^2)\, dx$. [variable sep not directly possible]

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy} = f(x,y)$$

$$\frac{dy}{dx} = \frac{(x^2 + 3y^2)/x^2}{(2xy)/x^2} = \frac{1 + 3\left(y/x\right)^2}{2\cdot\left(y/x\right)}$$

Let $\frac{y}{x} = v \Rightarrow y = vx$

Diff: $\frac{dy}{dx} = v + x\cdot\frac{dv}{dx}$

$$v + x\cdot\frac{dv}{dx} = \frac{1 + 3v^2}{2v} \longrightarrow \text{variable } (v, x)$$

$$x\cdot\frac{dv}{dx} = \frac{1 + 3v^2}{2v} - v$$

$$x\cdot\frac{dv}{dx} = \frac{1 + 3v^2 - 2v^2}{2v} = \frac{1+v^2}{2v}$$

Integrate: $\int \frac{2v}{1+v^2}\, dv = \int \frac{1}{x}\, dx \Rightarrow \ln\left|\frac{1+v^2}{x}\right| = c$

$\Rightarrow \ln\left|\frac{x^2+y^2}{x^3}\right| = c$

$\Rightarrow \frac{x^2+y^2}{x^3} = e^c = A$

$\ln\left|\frac{y^2}{x^3}\right| = c$ (×)

[c, A: arbitrary constant]

Q. Solve: $\left(1 + e^{x/y}\right) dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$.

$$e^{x/y}\left(1 - \frac{x}{y}\right) dy = -\left(1 + e^{x/y}\right) dx$$

$$\frac{dy}{dx} = \frac{-\left(1 + e^{x/y}\right)}{e^{x/y}\left(1 - \frac{x}{y}\right)} \Rightarrow \frac{dx}{dy} = \frac{e^{x/y}\left(1 - \frac{x}{y}\right)}{}$$

$$\left(\frac{dy}{dx}\right) = \frac{-(1+e^{x/y})}{e^{x/y}\left(1-\frac{x}{y}\right)} \Rightarrow \left(\frac{dx}{dy}\right) = \frac{e^{x/y}\left(1-\frac{x}{y}\right)}{-(1+e^{x/y})}$$

Let $\frac{x}{y} = v \Rightarrow x = vy$.

Diff: $dx = v\,dy + y\,dv$.

$$1 = v\cdot\frac{dy}{dx} + y\cdot\frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{1}{v}\left(1 - y\frac{dv}{dx}\right) \qquad (X)$$

Let $\frac{x}{y} = v \Rightarrow x = vy$.

Diff w.r.t y: $\frac{dx}{dy} = v + y\cdot\frac{dv}{dy}$.

$$v + y\cdot\frac{dv}{dy} = \frac{e^v(1-v)}{-(1+e^v)}$$

$$y\cdot\frac{dv}{dy} = \frac{e^v(v-1)}{1+e^v} - v$$

$$y\cdot\frac{dv}{dy} = \frac{\cancel{ve^v} - e^v - v - \cancel{ve^v}}{1+e^v} = -\frac{(v+e^v)}{1+e^v}$$

Integrate: $\int \frac{1+e^v}{v+e^v}\,dv = -\int\frac{1}{y}\,dy$

$$\left(\frac{x}{y} + e^{x/y}\right)y = c \Rightarrow x + y e^{x/y} = c.$$

Q. Solve: $x\cdot\frac{dy}{dx} - y = \sqrt{x^2+y^2}$

$$\Rightarrow \frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} = \left(\frac{y}{x}\right) + \sqrt{1+\left(\frac{y}{x}\right)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x} = \left(\frac{y}{x}\right) + \sqrt{1+\left(\frac{y}{x}\right)^2}$$

HW

Let $y = vx$.

Diff w.r.t x: