Continuity of Function $u = f(x,y)$:

A fn $u = f(x,y)$ is continuous at pt (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Note: If limit does not exist at pt (a,b), then fn will not be continuous.

Defn: A fn $f(x,y)$ is continuous at pt (a,b) if for any $\varepsilon > 0$.

s.t $|f(x,y) - \overbrace{f(a,b)}^{=l}| < \varepsilon$ whenever $\underbrace{(x-a)^2 + (y-b)^2 < \delta^2}_{\text{Circular neighbourhood around } (a,b)}$.

where $\delta = f(\varepsilon)$.

Q. $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$. Check continuity at pt $(0,0)$.

Start : $|f(x,y) - f(0,0)| < \varepsilon$.

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| < \varepsilon.$$

$$= \frac{|xy|}{\sqrt{x^2+y^2}} = \frac{|x|\cdot|y|}{\sqrt{x^2+y^2}}$$

$$\leq \frac{\sqrt{x^2+y^2}\cdot\cancel{\sqrt{x^2+y^2}}}{\cancel{\sqrt{x^2+y^2}}}$$

$$= \sqrt{x^2+y^2}.$$

$x^2 \geq 0$.

$x^2 + y^2 \geq y^2$.

+ve. sq. rt : $|y| \leq \sqrt{x^2+y^2}$

$y^2 \geq 0$.

$x^2 + y^2 \geq x^2$

+ve sq. rt : $|x| \leq \sqrt{x^2+y^2}$

$$\therefore \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leq \sqrt{x^2+y^2} < \varepsilon.$$

$\therefore \sqrt{x^2+y^2} < \varepsilon.$

sq. $\Rightarrow \quad x^2+y^2 < \varepsilon^2.$

If $\delta = \varepsilon$: $\boxed{x^2+y^2 < \delta^2} \Rightarrow$ δ-circular neighbourhood around $(0,0)$

$\therefore f(x,y)$ is continuous at $(0,0)$.

8. $f(x,y) = \begin{cases} x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right), & xy \neq 0. \\ 0 & , xy = 0. \end{cases}$ Check continuity at $(0,0)$

Start: $|f(x,y) - f(0,0)| < \varepsilon.$

$$\left| x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right) \right| < \varepsilon.$$

$$\leq \left| x\sin\left(\frac{1}{y}\right) \right| + \left| y\sin\left(\frac{1}{x}\right) \right| \quad (\because |a+b| \leq |a|+|b|)$$

$$= |x|\cdot\left|\sin\left(\frac{1}{y}\right)\right| + |y|\left|\sin\left(\frac{1}{x}\right)\right|$$

$$\leq |x| + |y| . \quad \left(\because \left|\sin\left(\frac{1}{y}\right)\right| \leq 1\right)$$

$$\leq \sqrt{x^2+y^2} + \sqrt{x^2+y^2} = 2\sqrt{x^2+y^2}.$$

$$\therefore \left| x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right) \right| \leq 2\sqrt{x^2+y^2} < \varepsilon.$$

$$\therefore 2\sqrt{x^2+y^2} < \varepsilon.$$

$$\Rightarrow \sqrt{x^2+y^2} < \left(\frac{\varepsilon}{2}\right)$$

$$\Rightarrow x^2+y^2 < \left(\frac{\varepsilon}{2}\right)^2$$

$f(x,y) = \begin{cases} x^2+2y, & (x,y) \neq (1,2) \\ 3, & (x,y) = (1,2) \end{cases}$

at $(x,y) \to (1,2)$ $\quad x^2+2y.$

$$\Rightarrow \quad x^2 + y^2 < \left(\frac{\varepsilon}{2}\right)^2 = \delta$$

$Lt_{(x,y)\to(1,2)} x^2 + 2y$

$= 1 + 4 = 5$

Let $\delta = \left(\frac{\varepsilon}{2}\right) \Rightarrow \boxed{x^2 + y^2 < \delta^2}$

$\therefore$ $f(x, y)$ is continuous at $(0,0)$.

Partial Derivatives

Eg: $u = f(x, y)$.

$\therefore \dfrac{\partial u}{\partial x} = f_x$ = Partial derivative of 'u' w.r.t x.
[Evaluate Δu due to Δx keeping y constant]

$\dfrac{\partial u}{\partial y} = f_y$ = Partial derivative of 'u' w.r.t 'y'
[Evaluate Δu due to Δy keeping 'x' constant].

Eg: $u(x,y) = x^2 - xy + 2y^2$.

$f_x = (2x - y) \Rightarrow f_x\big|_{(a,b)} = (2a - b)$

$f_y = (-x + 4y)$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} = f_{xx}.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} = f_{yy}.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\,\partial x} = f_{yx} = f_{xy}$$ [Young's Theorem].

Homogeneous functions:

$f(\lambda x, \lambda y) = \lambda^n f(x, y)$ ---- Function is homogeneous

$f(\lambda x, \lambda y) = \lambda^{n} f(x,y)$ ... Function is homogeneous of degree 'n' [λ = scalar]

Q. $f(x,y) = \dfrac{xy}{x^2+y^2}$.

$$f(\lambda x, \lambda y) = \frac{(\lambda x)(\lambda y)}{(\lambda x)^2 + (\lambda y)^2} = \frac{\lambda^2 \cdot xy}{\lambda^2(x^2+y^2)} = \frac{xy}{x^2+y^2}.$$

$\therefore \quad f(\lambda x, \lambda y) = \lambda^0 f(x,y)$ ... Homogeneous of degree zero.

Q. $f(x,y) = (ax+by)\sin\left(\dfrac{x}{y}\right), \; y \neq 0$.

$f(\lambda x, \lambda y) = \lambda^1 \cdot f(x,y)$ ... Homogeneous of degree 1.

$$= \{a(\lambda x) + b(\lambda y)\}\cdot \sin\left(\frac{\cancel{\lambda} x}{\cancel{\lambda} y}\right)$$

$$= \lambda \left[\{ax+by\}\cdot \sin\left(\frac{x}{y}\right)\right] = \lambda \cdot f(x,y)$$

Q. $f(x,y) = xy \cdot \dfrac{\sin(x^2+y^2)}{x^2+y^2}$.

$$f(\lambda x, \lambda y) = (\lambda x)(\lambda y) \cdot \frac{\sin\{(\lambda x)^2 + (\lambda y)^2\}}{(\lambda x)^2 + (\lambda y)^2}$$

$$= \frac{\cancel{\lambda^2} \cdot xy}{\cancel{\lambda^2}(x^2+y^2)} \sin\{\lambda^2(x^2+y^2)\}$$

$$= \frac{xy}{x^2+y^2} \cdot \sin\{\lambda^2(x^2+y^2)\}$$ ... Not Homogeneous

Euler's Theorem: If $u = f(x, y)$ is homogeneous of degree n, then: $\boxed{x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = nu}$

Q. $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$. $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} =$

(a) $\sin u$ (b) $\cos u$

(c) $\sin 2u$ (d) $\cos 2u$.

$$f(x, y) = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$$

$f(\lambda x, \lambda y) \neq \lambda^n \cdot f(x, y)$ ... Not homogeneous.

$$\boxed{z = \tan u} = \frac{x^3 + y^3}{x - y}.$$

$z = g(x, y) = \frac{x^3 + y^3}{x - y}$ ... Homogeneous of degree 2.

$\therefore$ Apply Euler's Th on z :-

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 2 \cdot (z).$$

HW. $x \cdot \frac{\partial}{\partial x}(\tan u) + y \cdot \frac{\partial}{\partial y}(\tan u) = 2(\tan u)$.

$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = ?$