Simple Linear Regression Model (SLRM).

Sample of size 'n' $(Y_i, X_i)_{i=1}^{n}$ is collected from the population.

True Model: $Y_i = \alpha + \beta X_i + u_i, \; i = 1, 2, \ldots, n$

α, β = unknown population parameters.

u_i = random disturbance term.

Estimated Model: $\hat{Y}_i = \hat{\alpha} + \hat{\beta} X_i, \; i = 1, 2, \ldots, n.$
(Fitted Model)

$\hat{\alpha}, \hat{\beta}$ = estimated values of α, β.
(computed based on sample obs)

$\hat{Y}_i$ = estimated Y_i.

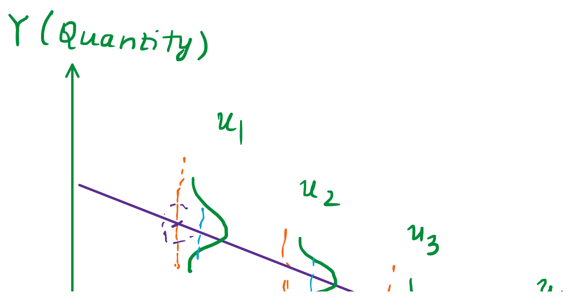
Logic of $\hat{Y}_i$:

Eg: $Y_i = 5, \; X_i = 3, \; \hat{Y}_i = 5.2$.

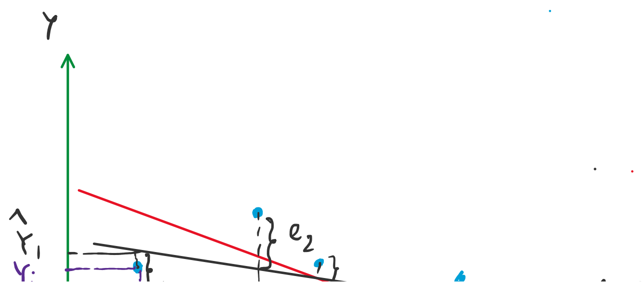
Y_i	X_i	$\hat{Y}_i$
5	3	5.2
9	6	8.7
12	7	11.9

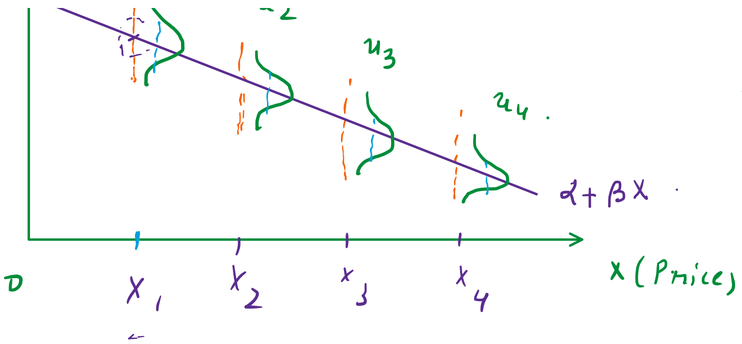
Hypothetically, if Y would have been influenced by X only, its value would have been 5.2, but as it is also influenced by the random factor u_i, its realized value is 5.

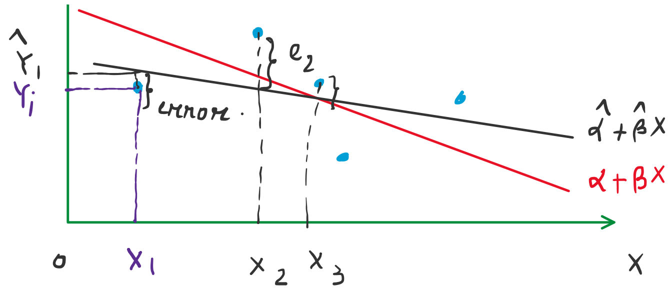
Population



Sample







Two types of errors:

(i) u_i = random disturbance term (unknown).

Since it is unknown, we will make some assumptions about it.

(ii) error in estimation $= e_i = Y_i - \hat{Y}_i$ (known)

e_i is known because it comes with fitting the model. Hence assumptions are not needed.

Assumptions of SLRM:

(Since u_i is random, we need to make assumption about the probability distribution of u_i).

(i) $E(u_i) = 0$. [Zero mean assumption]

(ii) $\text{cov}(X, u) = 0$ [No endogeneity]

(iii) $\text{Var}(u_i) = \sigma^2$ [Homogeneity / No heteroskedasticity].

(iv) $\text{cov}(u_i, u_j) = 0 \quad \forall\, i \neq j$ [No autocorrelation].

↳ (random disturbances are uncorrelated)

(iii) $\text{Var}(u_i) = \sigma^2$

$$\Rightarrow E(u_i^2) - [\underbrace{E(u_i)}]^2 = \sigma^2 \;\Rightarrow\; E(u_i^2) = \sigma^2 \quad \forall\, i$$

$$\Rightarrow E(u_i^2) - [\underbrace{E(u_i)}_{=0}]^2 = \sigma^2 \Rightarrow E(u_i^2) = \sigma^2 \quad \forall i$$

(iv) $cov(u_i, u_j) = 0$.

$$\Rightarrow E(u_i u_j) - \underbrace{E(u_i)}_{=0}\,\underbrace{E(u_j)}_{=0} = 0 \Rightarrow E(u_i u_j) = 0 \quad \forall i \neq j$$

Estimation of the Model

Method: OLS (Ordinary Least Square Method).

e_i = error in estimation = $Y_i - \hat{Y}_i$

In OLS, minimize: $\sum_{i=1}^{n} e_i^2$

Obtain $\hat{\alpha}, \hat{\beta}$ s.t $S = \sum_{i=1}^{n} e_i^2$ is minimized.

$$S = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta} X_i)^2 \quad (\because \hat{Y}_i = \hat{\alpha} + \hat{\beta} X_i)$$

For minimization:

$$\frac{\partial S}{\partial \hat{\alpha}} = 0 \Rightarrow \frac{\partial}{\partial \hat{\alpha}} \sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta} X_i)^2 = \sum_{i=1}^{n} \frac{\partial}{\partial \hat{\alpha}} (Y_i - \hat{\alpha} - \hat{\beta} X_i)^2$$

$$= \sum_{i=1}^{n} 2 \cdot (Y_i - \hat{\alpha} - \hat{\beta} X_i)(-1)$$

$$\frac{\partial S}{\partial \hat{\alpha}} = 0 \Rightarrow \sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta} X_i) = 0 \quad \text{..... (i)}$$

$$\frac{\partial S}{\partial \hat{\beta}} = 0 \Rightarrow \sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta} X_i) X_i = 0 \quad \text{..... (ii)}$$

(i) : $\sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} \hat{\alpha} - \hat{\beta} \sum_{i=1}^{n} X_i = 0$.

$$\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{n}{n}\hat{\alpha} + \hat{\beta}\frac{1}{n}\sum_{i=1}^{n} X_i \Rightarrow \bar{Y} = \hat{\alpha} + \hat{\beta}\bar{X} \quad \text{..... (ia)}$$

(ii) : $\sum_{i=1}^{n} Y_i X_i - \hat{\alpha} \sum_{i=1}^{n} X_i - \hat{\beta} \sum_{i=1}^{n} X_i^2 = 0 \Rightarrow \sum_{i=1}^{n} Y_i X_i = \hat{\alpha} \sum_{i=1}^{n} X_i + \hat{\beta} \sum_{i=1}^{n} X_i^2$ .... (iia)

Solve for $\hat{\alpha}$, $\hat{\beta}$ using eqns (ia) & (iia): -

use Cramer's Rule:

$$\hat{\beta} = \frac{\begin{vmatrix} 1 & \bar{Y} \\ \Sigma X_i & \Sigma Y_i X_i \end{vmatrix}}{\begin{vmatrix} 1 & \bar{X} \\ \Sigma X_i & \Sigma X_i^2 \end{vmatrix}}$$

HA. Expand $\hat{\beta}$.

Cramer's Rule

$a_1 x + b_1 y = c_1$

$a_2 x + b_2 y = c_2$

Coeff $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$