Given the production fn: $q = f(L, K)$, $w, r \rightarrow$ Parameters

For cost fn: $\underset{\{L,K\}}{\text{Min}} \quad wL + rK \quad \text{s.t} \quad \bar{q} = f(L, K)$

$\therefore$ We get: $C^* = wL^*(w, r, q) + rK^*(w, r, q)$

$= C^*(w, r, q)$

Deriving the production fn from the cost fn:-

$\therefore$ Given the cost fn $C^*(w, r, q)$. Find: $q = f(L, K)$.

(i) Use Shephard's Lemma: $\dfrac{\partial C^*}{\partial w} = L^*$, $\dfrac{\partial C^*}{\partial r} = K^*$

(ii) Using 2 eqns, eliminate w, r so that we get a relation b/w q, L, K .

Q. Cost fn: $C(w, r, q) = 2\sqrt{w \cdot r} \cdot q$. Find the production fn.

$$\left.\begin{aligned} \frac{\partial C}{\partial w} &= \cancel{2} \cdot \frac{1}{\cancel{2}\sqrt{w}} \cdot \sqrt{r} \cdot q = \sqrt{\frac{r}{w}} \cdot q = L^* \\ \frac{\partial C}{\partial r} &= \cancel{2} \cdot \frac{1}{\cancel{2}\sqrt{r}} \cdot \sqrt{w} \cdot q = \sqrt{\frac{w}{r}}\, q = K^* \end{aligned}\right\} \text{By Shephard's Lemma}$$

$$\left.\begin{aligned} \sqrt{\frac{r}{w}} \cdot q &= L^* \text{ ---- (i)} \\ \sqrt{\frac{w}{r}} \cdot q &= K^* \text{ ---- (ii)} \end{aligned}\right\} \text{Multiply: } \quad \begin{aligned} q^2 &= L \cdot K . \\ q &= L^{1/2} K^{1/2} \end{aligned}$$

Q. Cost fn: $C(w, r, q) = q \left[w^{\frac{\rho}{\rho-1}} + r^{\frac{\rho}{\rho-1}} \right]^{\frac{\rho-1}{\rho}}$, $\rho > 0$. Find the prodn fn.

$$\frac{\partial C}{\partial w} = \left(\cancel{\frac{\rho-1}{\rho}}\right) \cdot q \left[\cdots \right]^{\left(\frac{\rho-1}{\rho} - 1\right)} \cdot \cancel{\frac{\rho}{\rho-1}} \cdot w^{\left(\frac{\rho}{\rho-1} - 1\right)}$$

$$= q \cdot \left[\cdots \right]^{-1/\rho} \cdot w^{\frac{1}{\rho-1}}$$

$$\frac{\partial C}{\partial r} = q \left(\frac{\rho-1}{\rho}\right) \left[\cdots \right]^{\left(\frac{\rho-1}{\rho} - 1\right)} \cdot \frac{\rho}{\rho-1} \cdot r^{\left(\frac{\rho}{\rho-1} - 1\right)}$$

$$q \left[\cdots \right]^{-1/\rho} \cdot r^{\frac{1}{\rho-1}}$$

$$= q\,[\cdots]^{-1/\rho}\cdot r^{\frac{1}{\rho-1}}$$

$\therefore$ Now, Shephard's Lemma:

$$q\,[\cdots]^{-1/\rho}\cdot w^{\frac{1}{\rho-1}} = L^* \quad \cdots (i)$$
$$q\,[\cdots]^{-1/\rho}\cdot r^{\frac{1}{\rho-1}} = K^* \quad \cdots (ii).$$

$$(i):\ L^\rho = q^\rho\,[\cdots]^{-1}\cdot w^{\frac{\rho}{\rho-1}}$$
$$(ii):\ K^\rho = q^\rho\,[\cdots]^{-1}\cdot r^{\frac{\rho}{\rho-1}}$$

$$\therefore\ L^\rho + K^\rho = q^\rho\,\underbrace{[\cdots]}\,^{-1}\left\{\underbrace{w^{\frac{\rho}{\rho-1}} + r^{\frac{\rho}{\rho-1}}}\right\}$$

$$L^\rho + K^\rho = q^\rho\,\left[\cancel{w^{\frac{\rho}{\rho-1}} + r^{\frac{\rho}{\rho-1}}}\right]^{-1}\left\{\cancel{w^{\frac{\rho}{\rho-1}} + r^{\frac{\rho}{\rho-1}}}\right\}$$

$$L^\rho + K^\rho = q^\rho \Rightarrow q = [L^\rho + K^\rho]^{1/\rho}$$

Note: CES Production fn:

$$q = A\,[\delta L^{-\rho} + (1-\delta) K^{-\rho}]^{-1/\rho},\quad \delta > 0.$$

δ, ρ = Parameters of the CES production fn.

(i) If $\rho = -1 \Rightarrow q = A\,[\delta L + (1-\delta) K]$

$$= \underbrace{(A\delta)}_{a} L + \underbrace{\{A(1-\delta)\}}_{b} K = aL + bK$$

perfect substitutes.

(ii) If $\rho = 0 \Rightarrow q = A\,[\delta 1 + (1-\delta)\cdot 1]^{-1/0}$.

$$q = A\,[\cancel{\delta} + (1-\cancel{\delta})]^{-\infty} = A\cdot 1^{-\infty} \Rightarrow \text{Indeterminate Form}$$

Indeterminate Form: $\left[\frac{0}{0}\right], \left[\frac{\infty}{\infty}\right], \underbrace{0^0, \infty^0, 0^\infty, 1^\infty}_{\text{transform to } [\frac{0}{0}], [\frac{\infty}{\infty}]}, \ldots\ldots$

For solving Indeterminate Forms, we need L·Hopital's Rule.

Eg: $\lim_{x\to a} \frac{f(x)}{g(x)} \left[\frac{0}{0}\right] = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ [differentiate the num & deno separately]

Eg: $\lim_{x\to a} \frac{f(x)}{g(x)}$ $\left[\frac{0}{0}\right]$ $= \lim_{x\to a} \frac{f'(x)}{g'(x)}$ [differentiate the num & deno separately]

Continue with this process till we do not get any finite value.

Now, $q = A\left[\delta L^{-\rho} + (1-\delta)K^{-\rho}\right]^{-1/\rho}$. $[1^\infty$ for $\rho = 0]$

$$\frac{q}{A} = \left[\delta L^{-\rho} + (1-\delta)K^{-\rho}\right]^{-1/\rho}$$ $[1^\infty$ for $\rho = 0]$

$$\ln\left(\frac{q}{A}\right) = -\frac{1}{\rho}\cdot \ln\left[\delta L^{-\rho} + (1-\delta)K^{-\rho}\right]$$

$$\lim_{\rho\to 0} \ln\left(\frac{q}{A}\right) = -\lim_{\rho\to 0} \frac{\ln\left[\delta L^{-\rho} + (1-\delta)K^{-\rho}\right]}{\rho} \quad \left[\frac{0}{0}\right]$$

Using L·Hopital's Rule:-

$$= +\lim_{\rho\to 0} \frac{\frac{1}{\delta\cdot L^{-\rho} + (1-\delta)K^{-\rho}}\cdot\left[\cancel{-}\delta L^{-\rho}\cdot \ln L \cancel{-} (1-\delta)K^{-\rho}\ln K\right]}{1}$$

$$= \lim_{\rho\to 0} \frac{\delta L^{-\rho}\ln L + (1-\delta)K^{-\rho}\cdot\ln K}{\delta L^{-\rho} + (1-\delta)K^{-\rho}}$$

$$\lim_{\rho\to 0} \ln\left(\frac{q}{A}\right) = \frac{\delta\ln L + (1-\delta)\ln K}{\cancel{\delta} + (1-\cancel{\delta})} = \delta\ln L + (1-\delta)\ln K$$

$$= \ln L^{\delta} + \ln K^{(1-\delta)}$$

$$= \ln\left(L^{\delta}K^{1-\delta}\right)$$

$$\lim_{\rho\to 0} \ln\left(\frac{q}{A}\right) = \ln L^{\delta}K^{1-\delta}$$

$\therefore$ when $\rho \to 0$, $\ln\left(\frac{q}{A}\right) = \ln\left(L^{\delta}K^{1-\delta}\right)$

$$\boxed{q = A\cdot L^{\delta}K^{1-\delta}}$$ => Cobb Douglas Production fn.