Q. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation s.t $T(1,2) = (2,3)$, and $T(0,1) = (1,4)$, Then find: $T(5,6)$

(a) $(6,-1)$ (b) $(-6,1)$ (c) $(-1,6)$ (d) $(1,-6)$.

Defn: $T[c_1 \underset{\sim}{\alpha} + c_2 \underset{\sim}{\beta}] = c_1 T(\underset{\sim}{\alpha}) + c_2 T(\underset{\sim}{\beta})$.

$\underset{\sim}{\alpha} = (1,2)$ $\underset{\sim}{\beta} = (0,1)$.

$\therefore$ $T[\underbrace{c_1(1,2) + c_2(0,1)}] = c_1 T(1,2) + c_2 T(0,1)$

$= c_1(2,3) + c_2(1,4)$.

$\boxed{c_1(1,2) + c_2(0,1) = (5,6)}$ $\Rightarrow$ Find c_1, c_2.

$c_1 = 5$, $c_2 = -4$.

$(c_1, 2c_1) + (0, c_2) = (5,6)$

$\Rightarrow (c_1, 2c_1 + c_2) = (5,6)$

$\Rightarrow c_1 = 5$

$2c_1 + c_2 = 6 \Rightarrow c_2 = -4$.

(*) [RHS, $5(2,3) - 4(1,4) = (6,-1)$]

Q. For $n \in \mathbb{N}$, P_n denotes the vector space of all polynomials with degree $\leq n$. Define $T: P_n \to P_{n+1}$ by $T[p(x)] = p'(x) - \int_0^t p(t)\,dt$. Find the dimension of null space of T (Nullity) $\therefore$

$P_n = \{ p(x) = a_0 + a_1 x + \cdots + a_n x^n ;\ a_0, a_1, \ldots, a_n \text{ are scalars} \}$

Recap: linear trans $T: U \to V$.

$N(T) = \{ \underset{\sim}{\alpha} \in U : T(\underset{\sim}{\alpha}) = \underset{\sim}{0} \}$.

Here $N(T) = \{ p(x) \in P_n : T[p(\underset{\sim}{x})] = \underset{\sim}{0} \}$.

$p(x) \in P_n$ s.t $T[p(x)] = 0$.

$p'(x) - \int_0^t p(t)\,dt = 0$.

$p'(x) - \int_0^t p(t)\,dt = 0$.

$p'(x) = \int_0^t p(t)\,dt$ → Find $p(x)$

Diff: $p''(x) = p(x)$ ⇒

$N(T) = \phi$.

$\dim N(T) = 0$.

Eg: $n = 2$.

$p(x) = a_0 + a_1 x + a_2 x^2$

$p'(x) = a_1 + 2a_2 x$

$p''(x) = 2a_2$

$p''(x) = p(x)$.

$2a_2 = a_0 + a_1 x + a_2 x^2$

Compare coeff:

$a_2 = 0$

$a_1 = 0$

$a_0 = 2a_2 = 0$.

Fns: $f(x), g(x)$

$f\{g(x)\} \,/\, g\{f(x)\}$

Product of Linear Transformations:-

Consider 3 vector spaces U, V, W & 2 linear transformations as: $T_1 : U \to V$ and $T_2 : V \to W$, Then

$T_1 T_2 \,/\, T_2 T_1$ is defined as the product transformations

Note: Not necessary that both types of product transformations will be always defined and are equal.

$T_1 : U \to V$ and $T_2 : V \to W$.

$T_2 T_1 = T_2 [T_1(\underset{\sim}{u})] = T_2 [\underset{\sim}{v}] = \underset{\sim}{w}$. ; $\underset{\sim}{u} \in U, \underset{\sim}{v} \in V, \underset{\sim}{w} \in W$.

$T_1 T_2 = T_1 [T_2(\underset{\sim}{v})] = T_1 [\underset{\sim}{w}]$ --- Not possible .

Q. Let V be the vector space of all polynomials. Define linear transformations D & T as: $D[f(t)] = \dfrac{df(t)}{dt}$ and $T[f(t)] = \int_0^t f(t)\,dt$. Check $DT = TD$.

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots\cdots \quad , \quad f(t) \in V$$

$$DT = D\left[T(f(t))\right] = D\left[\int_0^t f(t)\,dt\right]$$

$$= D\left[\int_0^t (a_0 + a_1 t + a_2 t^2 + \cdots)\,dt\right]$$

$$= D\left[a_0 t + a_1 \frac{t^2}{2} + a_2 \frac{t^3}{3} + \cdots\right]$$

$$= \frac{d}{dt}\left[a_0 t + a_1 \frac{t^2}{2} + a_2 \frac{t^3}{3} + \cdots\right]$$

$$= a_0 + a_1 t + a_2 t^2 + \cdots = f(t) \checkmark$$

$$TD = T\left[D(f(t))\right] = T\left[\frac{d}{dt}\left\{a_0 + a_1 t + a_2 t^2 + \cdots\right\}\right]$$

$$= T\left[a_1 + 2a_2 t + \cdots\right]$$

$$= \int_0^t (a_1 + 2a_2 t + \cdots)\,dt$$

$$= (a_1 t + a_2 t^2 + \cdots\cdots) \checkmark \neq f(t)$$

DT & TD exist, but $DT \neq TD$.

Note: $D[T\{f(t)\}] = f(t)$ ---- Identity Transformation.

$$D[T\{f(t)\}] = I[f(t)]$$