Q. Let $f : (0, \infty) \to \mathbb{R}$ s.t $F(x) = \int_0^x f(t)\,dt$. If $F(x^2) = x^2(1+x)$ then $f(4) = ?$

(a) $5/4$ (b) 7 ~~(c)~~ 4 (d) 2.

$$F(x) = \int_0^x f(t)\,dt \Rightarrow \boxed{F'(x) = f(x)} \qquad f(4) = F'(4).$$

$$F(x^2) = x^2 + x^3.$$

$$F'(x^2) \cdot 2x = 2x + 3x^2$$

$$F'(x^2) = \frac{2x + 3x^2}{2x} = \frac{2+3x}{2} = f(x^2).$$

Put $x = 2 \Rightarrow f(4) = \frac{2+6}{2} = \frac{8}{2} = 4$.

Q. Let $f(x) = \max\{2-x, 2, 1+x\}$. Then $\int_{-1}^{1} f(x)\,dx =$

(a) 0 (b) 2 ~~(c)~~ $9/2$ (d) None.

$$f(x) = \begin{cases} 2-x, & -1 \le x < 0 \\ 2, & 0 \le x \le 1 \end{cases}$$

$$\therefore \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} (2-x)\,dx + \int_0^1 2\,dx = \frac{9}{2}$$

Q. Let $f(x) = a_0 \cos|x| + a_1 \sin|x| + a_2 |x|^3$. $f(x)$ is differentiable at $x = 0$ if: (a) $a_1 = 0,\ a_2 = 0$ ~~(c)~~ $a_1 = 0$

(b) $a_0 = 0$, $a_1 = 0$ (d) Any $a_0, a_1, a_2 \in \mathbb{R}$

$$f(x) = \begin{cases} a_0 \cos x + a_1 \sin x + a_2 x^3 , & x \geq 0 . \\ a_0 \cos x - a_1 \sin x - a_2 x^3 , & x < 0 . \end{cases}$$

Check diff LHD = RHD . at $x = 0$.

$$RHD\Big|_{x=0} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{a_0 \cos h + a_1 \sin h + a_2 \cdot h^3 - a_0}{h} =$$

$$LHD\Big|_{x=0} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{a_0 \cos h - a_1 \sin h - a_2 h^3 - a_0}{h}$$

$$=$$

$\Rightarrow a_1 = 0$ for LHD = RHD at $x = 0$.

Q. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ such that min $f(x) >$ max $g(x)$. Then :

(a) $|c| > \sqrt{2}$ (b) $0 < c < \frac{b}{2}$ (c) $|c| > \sqrt{2}|b|$ (d) None .

$f'(x) = 2x + 2b \Rightarrow f'(x) = 0 \Rightarrow x = -b . \Rightarrow$ Min $\quad f'' = 2$

$g'(x) = -2x - 2c \Rightarrow g'(x) = 0 \Rightarrow x = -c \Rightarrow$ Max . $\quad g'' = -2$.

min $f(x) = b^2 - 2b^2 + 2c^2 = 2c^2 - b^2$

max $g(x) = -c^2 + 2c^2 + b^2 = c^2 + b^2$.

$\therefore$ min $f(x) >$ max $g(x)$

$2c^2 - b^2 > c^2 + b^2$.

$c^2 > 2b^2 \Rightarrow |c| > \sqrt{2}\cdot|b|$.

Q. If $\boxed{2a+3b+6c=0}$, then the eqn $\boxed{ax^2+bx+c=0}$ has atleast one root in:

(a) $(0,1)$ (b) $(1,2)$ (c) $(-1,0)$ (d) $(2,3)$

$$f'(x) = ax^2+bx+c;$$

$$f(x) = \frac{ax^3}{3} + b\cdot\frac{x^2}{2} + cx$$

$f(x)$ is cont $[a,b]$
$f(x)$ is diff (a,b)
$\boxed{f(a)=f(b)}$.

$$\boxed{f(0)=0}, \quad \boxed{f(1)} = \frac{a}{3}+\frac{b}{2}+c = \frac{\boxed{2a+3b+6c}}{6} = 0.$$

$$f(0) = f(1)$$

$\therefore$ By Rolle's Th, $\exists$ a pt 'α' $\in (0,1)$ s.t. $f'(\alpha)=0$.

$\Rightarrow$ α is a root of $f'(x) = ax^2+bx+c$

$\alpha \in (0,1)$

Lagrange:

$$f'(c) = \frac{f(b)-f(a)}{b-a},$$

$c \in (a,b)$

Q. Let $y(t)$ be a soln to the differential eqn:

$(1+t)\cdot\frac{dy}{dt} - ty = 1$ and $y(0) = -1$. Then $y(1) = ?$

(a) $\frac{1}{2}$ (b) $e+\frac{1}{2}$ (c) $e-\frac{1}{2}$ (d) $-\frac{1}{2}$.

$$\frac{dy}{dt} - \left(\frac{t}{1+t}\right)\cdot y = \frac{1}{1+t}. \quad \text{---- Linear diff eqn.}$$

$$\therefore \text{ I.F } = e^{-\int \frac{t}{1+t}\,dt} = e^{-[t-\ln|1+t|]} = (1+t)\,e^{-t}.$$

$$\int \frac{t}{1+t}\,dt \Rightarrow \int \frac{t+1-1}{1+t}\,dt = \int 1 - \frac{1}{1+t}\,dt = t - \ln|1+t|$$

Diff eqn × I.F :

$$(1+t)\,e^{-t}\left[\frac{dy}{dt} + \frac{t}{1+t}\cdot y\right] = (1\cancel{+t})\,e^{-t}\cdot\frac{1}{(\cancel{1+t})}$$

$$\Rightarrow \frac{d}{dt}\left[y\cdot(1+t)e^{-t}\right] = e^{-t}$$

$$\Rightarrow \int d\left[y(1+t)\,e^{-t}\right] = \int e^{-t}dt$$

$$\Rightarrow y(1+t)\,e^{-t} = -e^{-t} + c.$$

Given $y(0) = -1 \Rightarrow$ $(-1)(1+0)\,e^{-0} = -e^{-0} + c$

$t=0,\ y=-1$ $\quad (-1) = (-1) + c \Rightarrow c = 0.$

$\therefore$ General soln : $y(1+t)\,e^{-t} = -e^{-t}$

$$y = -\frac{1}{1+t}$$

$$y(1) = -\frac{1}{2}$$

HW.

Q. Max : xy^2z^3 s.t $xyz \geq 0$ and $x+y+z = 3$.

(a) 1 (b) $\frac{1}{8}$ (c) $\frac{1}{4}$ (d) $\frac{27}{16}$.