Binomials

Question 1: If the sum of the coefficients of all even powers of x in the product $(1 + x + x^2 + \dots + x^{2n})(1 - x + x^2 - x^3 + \dots + x^{2n})$ is 61, then find the value of n.

2n+1 terms   2n+1 terms.

$$(1+x+x^2+\dots+x^{2n})(1-x+x^2-\dots+x^{2n}) = a_0x^0+a_1x^1+a_2x^2+\dots+a_{4n}x^{4n}.$$

given $a_0+a_2+a_4+\dots+a_{4n}=61$

put $x=1$.

$$(2n+1)(1) = a_0+a_1+a_2+a_3+\dots+a_{4n}.$$

$$2n+1 = a_0+a_1+a_2+a_3+\dots+a_{4n} \quad \text{— ①}$$

put $x=-1$.

$$(1)(2n+1) = a_0-a_1+a_2-a_3+\dots+a_{4n}$$

$$2n+1 = a_0-a_1+a_2-a_3+\dots+a_{4n} \quad \text{— ②}$$

Add ① and ②.

$$\cancel{2}(2n+1) = \cancel{2}(a_0+a_2+a_4+\dots+a_{4n})$$

$$2n+1 = 61$$

$$n = 30$$

Question 2: If α and β be the coefficients of x^4 and x^2 respectively in the expansion of $(x + \sqrt{(x^2 - 1)})^6 + (x - \sqrt{(x^2 - 1)})^6$, then:

(a) α + β = -30

(b) α − β = -132

(c) α − β = 60

(d) α + β = 60

$\alpha = -96$

$\beta = 36.$

$$(x+\sqrt{x^2-1})^6 + (x-\sqrt{x^2-1})^6.$$

$$a^3+b^3 = (a+b)^3 - 3ab(a+b)$$

$$a = (x+\sqrt{x^2-1})^2 \qquad b = (x-\sqrt{x^2-1})^2$$

$$a = x^2+x^2-1+2x\sqrt{x^2-1} \qquad b = x^2+x^2-1-2x\sqrt{x^2-1}$$

$$\boxed{a+b = 4x^2-2}.$$

$$ab = [(2x^2-1)+2x\sqrt{x^2-1}][(2x^2-1)-2x\sqrt{x^2-1}]$$

$$= (2x^2-1)^2 - [2x\sqrt{x^2-1}]^2$$

$$= 4x^4-4x^2+1 - 4x^2(x^2-1)$$

$$\boxed{ab = 1}$$

$$(a+b)^3 - 3ab(a+b) = (4x^2-2)^3 - 3(4x^2-2)$$

$$= 64x^6 - 96x^4 + 48x^2 - 8 - 12x^2 + 6.$$

$$= 64x^6 - 96x^4 + 36x^2 - 2.$$

Question 3: Find the coefficient of x^4 in the expansion of $(1 + x + x^2)^{10}$.

$n-r+r = n$

$(x^a+x^b+x^c)^n$. $(a+b)^n$ $T_{r+1} = {}^nC_r a^{n-r} b^r = \frac{n!}{r!(n-r)!} a^{n-r} b^r$

$\Rightarrow (1+x+x^2)^n$

Generic term $= \frac{n!}{p!\,q!\,r!} 1^p x^q (x^2)^r$

$\boxed{p+q+r=n}$ $n=10$

power of $x = q+2r = 4$

$\boxed{\begin{array}{l} q+2r=4 \\ p+q+r=10 \end{array}}$

$p/q/r \rightarrow$ integers ≥ 0.

p	q	r
6	4	0
7	2	1
8	0	2

① $\frac{10!}{6!\,4!}$ ② $\frac{10!}{7!\,2!\,1!}$ ③ $\frac{10!}{8!\,2!}$

$\frac{7\times 8\times 9\times 10}{24} = 210$ $\frac{8\times 9\times 10}{2} = 360$ $\frac{9\times 10}{2} = 45$

① + ② + ③ $= 615$

Question 4: If x is positive, the first negative term in the expansion of $(1 + x)^{27/5}$ is

$n = \frac{27}{5}$.

(a) 5th term (b) 6th term (c) 7th term (d) 8th term

$(a+b)^n$ n is a fraction / negative integer

$a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots\ldots$ infinite series

$T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} a^{n-r} b^r$.

$n-r+1 < 0$

$n+1 < r \rightarrow r > n+1 \rightarrow r > \frac{27}{5}+1$

$r > \frac{32}{5}$ $r > 6.4$ $\Rightarrow r = 7$

Question 7: If

$$a_n = \sqrt{7 + \left(\sqrt{7 + \sqrt{7 + \dots}}\right)}$$

having n radical signs then by methods of mathematical induction which is true

(a) $a_n < 7$ for all $n \geq 1$

(b) $a_n > 7$ for all $n \geq 1$

(c) $a_n > 3$ for all $n \geq 1$

(d) $a_n < 4$ for all $n \geq 1$

$$a_n = \sqrt{7 + a_n},$$

$$a_n^2 = a_n + 7$$

$$a_n^2 - a_n - 7 = 0$$

$$a_n = \frac{1 \pm \sqrt{1+28}}{2} = \frac{1 \pm \sqrt{29}}{2}$$

$$a_n = \frac{1+\sqrt{29}}{2}$$

$$5 < \sqrt{29} < 6$$

$$6 < 1+\sqrt{29} < 7$$

$$\boxed{3 < \frac{1+\sqrt{29}}{2} < 3.5}$$

Question 8: If the sum of the coefficients in the expansion of $(a + b)^n$ is 4096, then the greatest coefficient in the expansion is

(a) 1594 (b) 924 (c) 792 (d) 2924

Imp The value of $C_1^2 + C_2^2 .... + C_n^2$ (where C_i is the i^{th} coefficient of $(1 + x)^n$ expansion), is:

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$ ①

$(x+1)^n = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \dots + C_n$ ②

Multiply ① × ②

$(1+x)^{2n} = \boxed{C_0^2 x^n + C_1^2 x^n + C_2^2 x^n + \dots + C_n^2 x^n} + \dots$

Pick the term containing x^n from both sides.

$^{2n}C_n x^n = (C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2) x^n$

$^{2n}C_n = C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2$

$C_1^2 + C_2^2 + \dots + C_n^2 = \frac{(2n)!}{n!\,n!} - 1$

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, then $C_0C_1 + C_1 C_2 + ... + C_{n-1}C_n$ is equal to

(a) $\frac{(2n)!}{(n-1)!(n+1)!}$

(b) $\frac{(2n-1)!}{(n-1)!(n+1)!}$

(c) $\frac{(2n)!}{(n+2)!(n+1)!}$

(d) None of these

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n,$

$(x+1)^n = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \dots + C_n.$

Coeff of x^{n-1} from both sides.

$(1+x)^{2n} \rightarrow {}^{2n}C_{n-1}$