Real Analysis

Courses → They / Thy / 3hey

Real Analysis
Rai-Singhania

$\sqrt{2}$ Irr

Sup → max → Superior → Bara

Inf → mini → Inferior → chota

GLB LUB

$\frac{x+4}{x} = 1 + \frac{4}{x}$ $1 + \frac{4}{\infty}$

5 → Sup

3

2.33

1 → Inf

$u_x = x + \frac{4}{x}$

x ~ naturals

$-\infty, \ldots, 0, 1, 2 \ldots$

$u_1 = 1 + \frac{4}{1} = 1 + 4 = 5$

$u_2 = 2 + \frac{4}{2} = 4$

$u_3 = 3 + 4/3 = 4.33$

$u_\infty = \infty$

$\infty + \infty = 2\infty$

∞

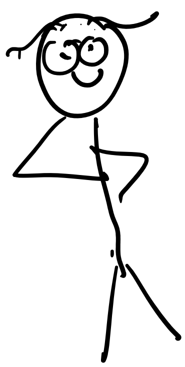
$\infty + \infty = \infty$

$\infty - \infty \neq 0$ → ∞ → UD

finite → UD

$i\infty = \infty$

$-\infty$ ∞



$-4 < x < 4$ — LUB

Rich ka Garib

Garib ka Rich

$-3 < x < 10$

$-2 < x < 100$

$1.3 < x < 97.33$

$-e^{7} < x < \log 10^{57}$

GLB

max in min

mins in max

max → Iphone 14 Pro Max 512GB/1024GB

Galaxy fold 5

Neighbourhood of a point

$x \to 0$

7800/month

Iphone 13

S22

$y = x^2$

$\frac{dy}{dx} = 2x$

$$L = (2x^2+3x)+4y$$

$x+y \le 3$

Constraint

CBSE

XI XII

Topology

GRF

Physics → Practical → Theory

$\sum$

$$\prod x_i = x_1 \cdot x_2 \cdots x_n$$

GIF → 2.99 → 2

2 → 2

2.07 → 2

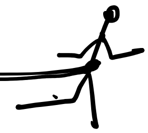
Pg class For 14.09.2023

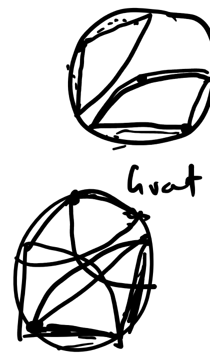
Bounded Convergent ⇒ (1)

1. Let $a_n = \sum_{k=1}^{n} \frac{n}{n^2 + k}$, for $n \in \mathbb{N}$. Then, the sequence $\{a_n\}$ is

(a) Convergent (b) Bounded but not convergent
(c) Diverges to ∞ (d) Neither bounded nor diverges to ∞

Bound





$\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}$ --- (i)

$\frac{n}{n^2+1} < \frac{n}{n^2}$ --- (ii)

∀ v.g.

$a_n = \frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}$

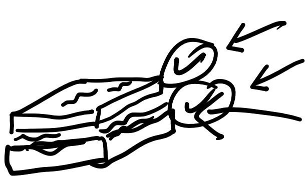
$< \frac{n}{n^2} + \frac{n}{n^2} + \cdots + \frac{n}{n^2} = \frac{n^2}{n^2} = 1$

$\boxed{a_n < 1}$ ∀ $n \in N$ $a_n > 0$ ∀ $n \in N$

B/A + B/B ∃ there exists

Now, $k \le n$ $n^2 + k \le n^2 + n$

$\frac{n}{n^2+k} \geq \frac{n}{n^2+n}$



$a_n < 1$, $a_n > 0$ $a_n = \sum \frac{n}{n^2+k} \geq \sum \frac{n}{n^2+n}$

$\lim \sum \frac{n}{n^2} \geq a_n \geq \sum \frac{n}{n^2+n}$

$\Rightarrow \frac{n \cdot n}{n^2} \geq a_n \geq \frac{n \cdot n}{n^2+n}$

$\lim_{n\to\infty} \frac{n}{n+1} \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty} 1$

$\boxed{1 \le \lim_{n\to\infty} a_n \le 1}$ $\Rightarrow \frac{n}{n+1} \le a_n \le 1$

2. The number of real roots of the equation $x^3 + x - 1 = 0$ is
(a) 0 (b) 1 (c) 2 (d) 3

2. The number of real roots of the equation $x^3 + x - 1 = 0$ is

(a) 0 (b) 1 (c) 2 (d) 3

$x^3 + x - 1$ (1) (+ve Real)

$-x^3 - x - 1$ (0)

$3 - 1 = 2$ Thing

$\lim_{n\to\infty} a_n = 1$

$f_1 \; f_2 \; f_3$ Interval

Lt

3. The value of $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+kn}}$ is

(a) $2(\sqrt{2}-1)$ (b) $2\sqrt{2}-1$ (c) $2-\sqrt{2}$ (d) $\frac{1}{2}(\sqrt{2}-1)$

Riemann's Sum for def int. $\lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} = \int_0^1 f(x)\,dx$

$= \lim_{n\to\infty} \frac{1}{n\sqrt{1+\frac{k}{n}}} = \lim_{n\to\infty}\sum \frac{1}{n}\frac{1}{\sqrt{1+\frac{k}{n}}} = \int_0^1 \frac{dx}{\sqrt{1+x}}$

$= \lim_{n\to\infty}\sum \frac{1}{\sqrt{n^2+kn}} = 2(1+x)^{1/2}\Big|_0^1 = 2(\sqrt{2}-1)$

4. *The set of all x at which the power series* $\sum_{n=1}^{\infty} \frac{n}{(2n+1)^2}(x-2)^{3n}$ *converges, is*

(a) [-1,1) (b) [-1,1] (c) [1,3) (d) [1,3]

5. *Consider the following subsets of* $\mathbb{R}$:

$E = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}, F = \left\{ \frac{1}{1-x} : 0 \le x < 1 \right\}$ *Then*

(a) Both E and F are closed (b) E is closed and F is not closed
(c) E is NOT closed and F is closed (d) Neither E nor F is closed.

(c) E is NOT closed and F is closed (d) Neither E nor F is closed.

$E = 1/2, 2/3, 3/4, 4/5 \dots \Rightarrow 0.5, 0.67, 0.75, 0.80 \longrightarrow 1$

$\frac{1}{2} \le x < 1 \ \forall x \in E \rightarrow E$ is Bounded

$1 \notin E$ ($\because$ if $1 \in E \Rightarrow \frac{n}{n+1} = 1 \Rightarrow n = n+1 \Rightarrow 1 = 0$ X)

But E contains infinitely many values in the neighbourhood of 1

So, E is Not closed.

$F = \frac{1}{1-x}; 0 \le x < 1$

As, $0 \le x < 1 \Rightarrow -1 < -x \le 0$

$0 < 1-x \le 1$

$1-x \le 1 \Rightarrow \frac{1}{1-x} \ge 1$

F is closed

6. (a) *Let $\{a_n\}$ be a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} a_n$ converges, and let $\{k_n\}$ be a strictly increasing sequence of positive integers. Show that $\sum_{n=1}^{\infty} a_{k_n}$ also converges.*

(b) Suppose $f:[0,1] \to \mathbb{R}$ is differentiable and $f'(x) \leq 1$ at every $x \in (0,1)$. If $f(0) = 0$ and $f(1)=1$, show that $f(x)=x$ for all $x \in [0,1]$.

Real Analysis 21 Sep 2023

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Sufficient test..

7. Show that the series $\sum_{n=1}^{\infty} \frac{x}{\sqrt{n}(1+n^p x^2)}$ converges on $\mathbb{R}$ for $p > 1$

$f_n = \frac{x}{\sqrt{n}(1+n^p x^2)}$

$f_n'(x) = \frac{1}{\sqrt{n}}\left[\frac{(1+n^p x^2)\cdot 1 - x(2n^p \cdot x)}{(1+n^p x^2)^2}\right]$

$= \frac{1}{\sqrt{n}}\left[\frac{1-n^p x^2}{(1+n^p x^2)^2}\right]$

Sup @ $f_n'(x) = 0$

$1 - n^p x^2 = 0 \Rightarrow x = \frac{1}{n^{p/2}}$

$\sup f_n(x) = \frac{1/n^{p/2}}{\sqrt{n}\left[1 + n^p \cdot \frac{1}{n^p}\right]} = \frac{1}{2n^{\frac{p+1}{2}}}$

$p>1$
$p<1$
$p=1$
$p>0$

$\frac{4+n}{n} \to \left(\frac{4}{n}+1\right)$

$n \to \infty$ N

$\frac{4}{1}+1$, $4/2+1$, $4/3+1$

$\Rightarrow$ 5, 3, 2.33, 2, 1.8, 1.66 1

GLB

LUB

Sup $\to$ 5

Inf $\to$ 1

kappa

K K

$f_n(x) \le \sup f_n(x) \ \forall n \in N$, $x \in R$

$\Rightarrow |f_n(x)| \le \frac{1}{2n^{\frac{p+1}{2}}} < \frac{1}{n^{\frac{p+1}{2}}}$

p-test $\to \sum_{n=1}^{\infty} \frac{1}{n^{\frac{p+1}{2}}}$ Converges if $\frac{p+1}{2} > 1$ $p>1$

Comparison test $\to \sum f_n(x)$ Conv $\forall x \in R$ If $p>1$

8. (a) If E is a subset of $\mathbb{R}$ that does not contain any of its limit points, then prove that E is a countable set.

(b) Let $f : (a,b) \to \mathbb{R}$ be a continuous function. If f is uniformly continuous, then prove that there exists a continuous function $g : [a,b] \to \mathbb{R}$, such that $g(x) = f(x)$ for all $x \in (a,b)$.

9. Let $\{x_n\}$ be the sequence $+\sqrt{1}, -\sqrt{1}, +\sqrt{2}, -\sqrt{2}, +\sqrt{3}, -\sqrt{3}, +\sqrt{4}, -\sqrt{4}, \ldots$ If $y_n = \dfrac{x_1 + x_2 + \ldots + x_n}{n}$ for all $n \in \mathbb{N}$, then the sequence $\{y_n\}$ is

Conv.

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$y_n = \dfrac{x_1 + \cdots x_n}{n}$

$x_n = -x_{n-1} \;\forall\; n \in \mathbb{N}$ n is even

$y_n = 0 \;\forall$ even (n)

If n is odd $y_n = \dfrac{1}{n}\sqrt{\dfrac{n+1}{2}} \Rightarrow \lim_{n\to\infty} y_n = 0$

(a) monotonic
(b) not bounded
(c) bounded but not convergent
(d) convergent

10. The number of distinct real roots of the equation $x^9 + x^7 + x^5 + x^3 + x + 1 = 0$ is

(a) 1 (b) 3 (c) 5 (d) 9

+ve =0

- - - - - +

11. *If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = 3$, then the series $\sum_{n=0}^{\infty} a_n x^n$.*

(a) Converges absolutely for $x = -2$ (b) Converges but not absolutely for $x = -1$

(c) Converges but not absolutely for $x = 1$ (d) Diverges for $x = -2$

Radius of Conv R ≥ 3 $\boxed{R \geq 3}$

The series Converges absolutely inside the region of Convergence.

Series Converges absolutely for $x = -1, -2, 1$

$$\left|\frac{x}{1+|x|}\right| = \frac{|x|}{|1+|x||} = \frac{|x|}{1+|x|}$$

$$= 1 - \frac{1}{1+|x|} < 1$$

$$\left[\frac{1}{1+|x|} > 0\right]$$

12. If $Y = \left\{ \frac{x}{1+|x|} \mid x \in \mathbb{R} \right\}$, then the set of all limit points of Y is

(a) (-1, 1) (b) (-1, 1] (c) [0, 1] (d) [-1, 1]

Not A

But D

So, $\Rightarrow \left|\frac{x}{1+|x|}\right| < 1 \;\forall\; x \in R$

$< 1 \;\forall\; x \in R$

$\Rightarrow -1 < \frac{x}{1+|x|}$

$Y = (-1, 1)$

13. ***(a)*** *Examine whether the following series is convergent* $\sum_{n=1}^{\infty} \frac{n!}{1 \cdot 3 \cdot 5 \ldots (2n-1)}$

(b) *For each* $x \in \mathbb{R}$*, let* $[x]$ *denotes the integer less than or equal to x. Further, for a fixed* $\beta \in (0,1)$*, define* $a_n = \frac{1}{n}[n\beta] + n^2\beta^n$ *for all* $n \in \mathbb{N}$*. Show that the sequence* $\{a_n\}$ *converges to* β*.*

14. (a) Show that the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2$ for $x \in \mathbb{R}$, is not uniformly continuous.

(b) For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. If the sequence $\{f_n\}$ converges uniformly on $\mathbb{R}$ to a function $f : \mathbb{R} \to \mathbb{R}$, then show that f is uniformly continuous.

15. **(a)** *Let A be a nonempty bounded subset of* $\mathbb{R}$*. Show that* $\{x \in \mathbb{R} \mid x \geq a \text{ for all } a \in A\}$ *is a closed subset of* $\mathbb{R}$*.*

(b) *Let* $\{x_n\}$ *be a sequence in* $\mathbb{R}$ *such that* $|x_{n+1} - x_n| < \frac{1}{n^2}$ *for all* $n \in \mathbb{N}$*. Show that the sequence* $\{x_n\}$ *is convergent.*

16. *Let A and B be subsets of $\mathbb{R}$. Which of the following is NOT necessarily true?* ***(MCQ)***

(a) $(A \cap B)^{\circ} \subseteq A^{\circ} \cap B^{\circ}$

(b) $A^{\circ} \cup B^{\circ} \subseteq (A \cup B)^{\circ}$

(c) $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$

(d) $\overline{A} \cap \overline{B} \subseteq \overline{A \cap B}$

GIF

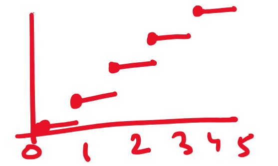
17. Let [x] denote the greatest integer function of x. The value of α for which the function

$$f(x)=\begin{cases}\dfrac{\sin[-x^2]}{[-x^2]}, & x\neq 0\\ \alpha, & x=0\end{cases}$$

is continuous at x = 0 is (MCQ)

(a) 0 (b) sin(-1) (c) sin 1 (d) 1

$[2.57] \to 2$



step function

$[-1 \cdot h^2] = (-1)$

LHL = RHL

LHL → $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{\sin[-h^2]}{[-h^2]} = \lim_{h\to 0} \frac{\sin(-1)}{(-1)}$

$= -\sin(-1) = \sin 1$

RHL → $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{\sin[-h^2]}{[-h^2]} = \sin 1$

f is cont @ x = 0

$\alpha = \sin 1$

18. Let the function f(x) be defined by $f(x)=\begin{cases} e^{x}, & x \text{ is rational} \\ e^{1-x}, & x \text{ is irrational} \end{cases}$ for x in (0, 1). Then **(MCQ)**

(a) f is continuous at every point in (0, 1).
(b) f is discontinuous at every point in (0, 1).
(c) f is discontinuous only at one point in (0, 1).
(d) f is continuous only at one point in (0, 1).

The interval (0,1) → ∞ no of rational + irrational numbers.
Hence, f is continuous for x ∈ (0,1) then $e^{x} = e^{1-x}$

Hence, $x = 1 - x$ $2x = 1$ $x = 1/2$

f(x) is cont @ $x = 1/2$

only 1 pt (0,1)

19. *Let* $x_n = \left(1-\frac{1}{3}\right)^2\left(1-\frac{1}{6}\right)^2\left(1-\frac{1}{10}\right)^2 \cdots \left(1-\frac{1}{\frac{n(n+1)}{2}}\right)^2$, $n \geq 2$. *Then* $\lim\limits_{n\to\infty} x_n$ *is* **(MCQ)**

(a) $\frac{1}{3}$ (b) $\frac{1}{9}$ (c) $\frac{1}{81}$ (d) 0

20. *The function to which the power series* $\sum_{n=1}^{\infty} (-1)^{n+1} n\, x^{2n-2}$ *converges is* __________. **(NAT)**

21. *Let* $0 < a \leq 1, s_1 = \frac{a}{2}$ *and for* $n \in \mathbb{N}$, *let* $s_{n+1} = \frac{1}{2}(s_n^2 + a)$. *Show that the sequence* $\{s_n\}$ *is convergent and find its limit.*

22. *Let K be a compact subset of* $\mathbb{R}$ *with non-empty interior. Prove that, K is of the form* $[a, b]$ *or of the form* $[a, b] \setminus \cup I_n$, *where* $\{I_n\}$ *is a countable disjoint family of open intervals with end points in K.*

23. *The coefficient of $(x-1)^2$ in the Taylor series expansion of $f(x) = xe^x$ $(x \in \mathbb{R})$ about the point $x = 1$ is* ***(MCQ)***

(a) $\frac{e}{2}$ (b) $2e$ (c) $\frac{3e}{2}$ (d) $3e$

3\. The radius of convergence of the power series $\sum_{n=0}^{\infty} 2^{2n} x^{n^2}$ is **(MCQ)**

(a) $\frac{1}{4}$ (b) 1 (c) 2 (d) 4

Let, $n^2 = N$

$n = \sqrt{N}$

$\sum 2^{2n} \cdot x^{n^2} = \sum 2^{2\sqrt{N}} \cdot x^{N}$

$\frac{1}{R} = \lim_{N\to\infty} |a_N|^{1/N}$

Here, $a_N = 2^{2\sqrt{N}}$

$\lim_{N\to\infty} \left|2^{2\sqrt{N}}\right|^{1/N} = \lim_{N\to\infty} \left|2^{\frac{2}{\sqrt{N}}}\right| = 1$

$\frac{1}{R} = 1$ $\boxed{R = 1}$

25. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $x + \int_0^x f(t)dt = e^x - 1$ for all $x \in \mathbb{R}$. Then the set $\{x \in \mathbb{R} : 1 \le f(x) \le 2\}$ is the interval (MCQ)

(a) [log 2, log 3] (b) [2 log 2, 3 log 3] (c) $[e - 1, e^2 - 1]$ (d) $[0, e^2]$

Us and

Diff b/w → $1 + f(x) = e^x$

$f(x) = e^x - 1$

Now, $1 \le f(x) \le 2$

$1 \le e^x - 1 \le 2$

$2 \le e^x \le 3$

$\ln 2 \le x \le \ln 3$

Solve as limit problem..

$$x_n = 2^{2n} \cdot 2\sin^2\left(\frac{1}{2^{n+1}}\right) \quad [1-\cos 2\theta = 2\sin^2\theta]$$

26. Let $x_n = 2^{2n}\left(1-\cos\left(\frac{1}{2^n}\right)\right)$ for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ **(MCQ)**

(a) does NOT converge
(b) converges to 0
(c) converges to $\frac{1}{2}$
(d) converges to $\frac{1}{4}$

$$x_n = \frac{2^{2n+2}}{2}\sin^2\left(\frac{1}{2^{n+1}}\right)$$

$$\lim_{n\to\infty} x_n = \frac{1}{2}\lim_{n\to\infty}\frac{\sin^2\left(\frac{1}{2^{n+1}}\right)}{\frac{1}{2^{2n+2}}}$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(\frac{\sin\left(\frac{1}{2^{n+1}}\right)}{\frac{1}{2^{n+1}}}\right)^2$$

$$\Rightarrow \frac{1}{2}[1]^2 = \frac{1}{2}$$

CMI

27. Let $\{x_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty}(x_{n+1}-x_n) = c$, where c is a positive real number. Then, the sequence $\left\{\frac{x_n}{n}\right\}$ **(MCQ)**

(a) is NOT bounded ×
(b) is bounded but NOT convergent ×
(c) converges to c
(d) converges to 0 ×

$$\lim (n+1-n) = c \quad [1 = c]$$

$\lim_{n\to\infty} (n+1 - n) = c$ $\boxed{1 = c}$

Now, $\left\{\frac{x_n}{n}\right\} = \left(\frac{n}{n}\right) = \{1\}$

Since Const Seqnc is Bounded ✗

(a) → Since Const Seqnc is Bounded ✗

(b) → Const → Convt ✗

(c) $\left\{\frac{x_n}{n}\right\} = \{1\} \Rightarrow \lim_{n\to\infty} \left\{\frac{x_n}{n}\right\} = 1 = c$

(d) →

$a_{n+1} = \frac{(-1)^{n+1}\cdot(n+1)}{2^{n+1}} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n}\right| = \left|\frac{n+1}{n}\cdot\frac{1}{2}\right|$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} < 1 \Rightarrow \sum a_n$ Con absolutely

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28. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series, where $a_n = \frac{(-1)^n n}{2^n}, b_n = \frac{(-1)^n}{\log(n+1)}$ for all $n \in \mathbb{N}$. Then **(MCQ)**

✗ (a) both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent.

✓ (b) $\sum_{n=1}^{\infty} a_n$ is absolutely convergent but $\sum_{n=1}^{\infty} b_n$ is conditionally convergent.

✗ (c) $\sum_{n=1}^{\infty} a_n$ is conditionally convergent but $\sum_{n=1}^{\infty} b_n$ is absolutely convergent.

✗ (d) both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are conditionally convergent.

$|b_n| = \frac{1}{\ln(n+1)}$ $\sum|b_n| = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \dots > \frac{1}{2} + \frac{1}{3} + \dots$

[As $\ln x < x$ so; $\frac{1}{\ln x} > \frac{1}{x}$]

Radar
Noon
Level
Civic

Reviver

Kayak

Madam

Live → [evil]

Semordnilaps

Drawer

$|b_n| \geq \sum \frac{1}{n}$ diverge

$(|b_n|) \to$ Bound

$\sum b_n = \sum \frac{(-1)^n}{\ln(n+1)}$

$\sum b_n = \sum \frac{(-1)^n}{\ln(n+1)}$ $n \to \infty$

$\frac{1}{\ln(n+1)}$ $\downarrow$ $\frac{1}{\ln(n+1)} \to 0$

$\sum b_n \to$ Conv by Leibnitz test..

$\sum b_n \to$ Conditionally Conv

29. *The set* $\left\{ \frac{x^2}{1+x^2} : x \in \mathbb{R} \right\}$ *is* **(MCQ)**

(a) *connected but NOT compact in* $\mathbb{R}$ ✓
(b) *compact but NOT connected in* $\mathbb{R}$
(c) *compact and connected in* $\mathbb{R}$
(d) *neither compact nor connected in* $\mathbb{R}$

$x \in R$ $x^2 \geq 0$ $1+x^2 \geq 0$

$\frac{x^2}{1+x^2} \geq 0$ $\forall x \in R$

$\frac{x^2}{1+x^2} < 1$ $\frac{x^2}{1+x^2} \neq 1$

[$\because$ If $\frac{x^2}{1+x^2} = 1$
$x^2 = 1 + x^2$
$\Rightarrow 1 = 0$] ✗

$S = [0, 1)$

S is bounded but not closed

Not Compact

S is Connected

30. *The set of all limit points of the set* $\left\{ \frac{2}{x+1} : x \in (-1, 1) \right\}$ *in* $\mathbb{R}$ *is* **(MCQ)**

(a) $[1, \infty)$ ✓ (b) $(1, \infty)$ (c) $[-1, 1]$ (d) $[-1, \infty)$

$-1 < x < 1$
$0 < x+1 < 2$ $< \infty$
$\frac{1}{2} < \frac{1}{x+1}$ $1 < \frac{2}{x+1} < \infty$
$S = (1, \infty)$

$[1, \infty)$

4th

31. Let $S = [0, 1] \cup [2, 3)$ and let $f : S \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 2x & \text{if} \quad x \in [0,1] \\ 8 - 2x & \text{if} \quad x \in [2,3) \end{cases}$

If $T = \{f(x) : x \in S\}$, then the inverse function $f^{-1} : T \to S$ **(MCQ)**

(a) does NOT exist
(b) exists and is continuous
(c) exists and is NOT continuous
(d) exists and is monotonic

32. Let $f(x) = x^3+x$ and $g(x) = x^3 - x$ for all $x \in \mathbb{R}$. If f^{-1} denotes the inverse function of f, then the derivative of the composite function $g \circ f^{-1}$ at the point 2 is **(MCQ)**

(a) $\frac{2}{13}$ (b) $\frac{1}{2}$ (c) $\frac{11}{13}$ (d) $\frac{11}{4}$

Let, $x^2 = y$

$f'(y) = 1 - y^{3/2}$

$f(y) = y - 2/5\, y^{5/2} + C$

33. Let $f: (0, \infty) \to \mathbb{R}$ be a differentiable function such that $f'(x^2) = 1 - x^3$ for all $x > 0$ and $f(1) = 0$. Then, $f(4)$ equals **(MCQ)**

(a) $\frac{-47}{5}$ (b) $\frac{-47}{10}$ (c) $\frac{-16}{5}$ (d) $\frac{-8}{5}$

$f(1) = 0$

$0 = 1 - 2/5 + C$

$C = -3/5$

$f(4) = 4 - \frac{2}{5}(4)^{5/2} - \frac{3}{5}$

$= 4 - \frac{2}{5}(32) - \frac{3}{5}$

$= \frac{-47}{5}$

(B) is Corr..

$x^6 - x^5 \le 100$

$T = \{x^2 - 2x,\ x \in (0, \infty)\}$

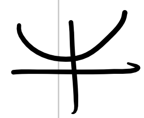
let, $y = x^2 - 2x$ $\quad y + 1 = (x-1)^2$ upward parabola

vertex $(-1, 1)$

34. Let $S = \{x \in \mathbb{R} : x^6 - x^5 \le 100\}$ and $T = \{x^2 - 2x : x \in (0, \infty)\}$. The set $S \cap T$ is *(MCQ)*

(a) closed and bounded in $\mathbb{R}$ (b) closed but NOT bounded in $\mathbb{R}$

(c) bounded but NOT closed in $\mathbb{R}$ (d) neither closed nor bounded in $\mathbb{R}$



In the set S let, $u = x^6 - x^5 - 100$

$\frac{du}{dx} = 6x^5 - 5x^4 = 0$

$x^4(6x - 5) = 0$

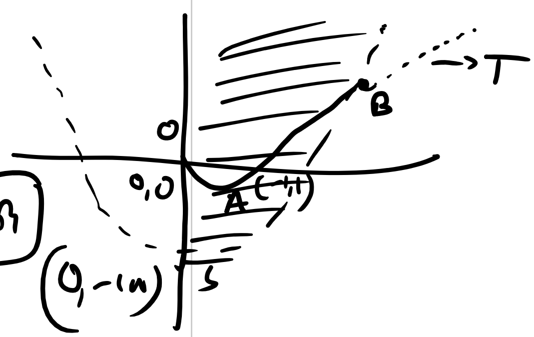
$x = 0, 5/6$

Again re-diff $\frac{d^2u}{dx^2} = 30x^4 - 20x^3$

@ $x = 5/6$ $\frac{d^2y}{dx^2} > 0$ $\quad$ u is min @ $x = 5/6$

@ $x = 0$, $u = -100$

$u \to$ upw. sloping func



$S \cap T$ Contain all pts 0, A, B

closed + Bound in $\mathbb{R}$

35. Let $f : (0,1) \to \mathbb{R}$ be a differentiable function such that $|f'(x)| \le 5$, for all $x \in (0,1)$. Show that the sequence $\left\{ f\left(\frac{1}{n+1}\right) \right\}$ converges in $\mathbb{R}$.

36. *If K is a non-empty closed subset of* $\mathbb{R}$*, then show that the set* $\{ x + y : x \in K, y \in [1,2] \}$ *is closed in* $\mathbb{R}$.

37. *Let S be a nonempty subset of $\mathbb{R}$. If S is a finite union of disjoint bounded intervals, then which one of the following is true?* ***(MCQ)***

(a) *If S is not compact, then* $\sup S \notin S$ *and* $\inf S \notin S$

(b) *Even if* $\sup S \in S$ *and* $\inf S \in S$, *S need not be compact*

(c) *If* $\sup S \in S$ *and* $\inf S \in S$, *then S is compact*

(d) *Even if S is compact, it is not necessary that* $\sup S \in S$ *and* $\inf S \in S$

38. Let $\{x_n\}$ be a convergent sequence of real numbers. If $x_1 > \pi + \sqrt{2}$ and $x_{n+1} = \pi + \sqrt{x_n - \pi}$ for $n \geq 1$, then which one of the following is the limit of this sequence? *(MCQ)*

(a) $\pi + 1$ (b) $\pi + \sqrt{2}$ (c) π (d) $\pi + \sqrt{\pi}$

39. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 0$. If for all $x \in \mathbb{R}$, $1 < f'(x) < 2$, then which one of the following statements is true on $(0, \infty)$? *(MCQ)*

(a) f is unbounded (b) f is increasing and bounded
(c) f has at least one zero (d) f is periodic

40. *Let A be a nonempty subset of* $\mathbb{R}$. *Let I (A) denote the set of interior points of A. Then I(A) can be* **(MCQ)**

(a) empty
(b) singleton
(c) a finite set containing more than one element
(d) countable but not finite

41. *The limit* $\lim_{x \to 0^+} \frac{1}{\sin^2 x} \int_{\frac{x}{2}}^{x} \sin^{-1} t \, dt$ *is equal to* ***(MCQ)***

(a) 0 *(b)* $\frac{1}{8}$ *(c)* $\frac{1}{4}$ *(d)* $\frac{3}{8}$

42. *Let* $S = \bigcap_{n=1}^{\infty}\left(\left[0, \frac{1}{2n+1}\right] \bigcup \left[\frac{1}{2n}, 1\right]\right)$. *Which one of the following statements is FALSE?* **(MCQ)**

(a) *There exist sequences* $\{a_n\}$ *and* $\{b_n\}$ *in [0, 1] such that* $S = [0,1] \setminus \bigcup_{n=1}^{\infty}(a_n, b_n)$

(b) *[0, 1]\S is an open set*

(c) *If A is an infinite subset of S, then A has a limit point*

(d) *There exists an infinite subset of S having no limit points*

43. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a strictly increasing continuous function. If* $\{a_n\}$ *is a sequence in* $[0, 1]$*, then the sequence* $\{f(a_n)\}$ *is* **(MCQ)**

(a) increasing (b) bounded (c) convergent (d) not necessarily bounded

44. *Which one of the following statements is true for the series* $\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{n^{2n}}$? ***(MCQ)***

(a) The series converges conditionally but not absolutely
(b) The series converges absolutely
(c) The sequence of partial sums of the series is bounded but not convergent
(d) The sequence of partial sums of the series is unbounded

45. *The sequence* $\left\{\cos\left(\frac{1}{2}\tan^{-1}\left(-\frac{n}{2}\right)^{n}\right)\right\}$ *is* **(MCQ)**

(a) monotone and convergent

(b) monotone but not convergent

(c) convergent but not monotone

(d) neither monotone nor convergent

46. *Let G and H be nonempty subsets of $\mathbb{R}$, where G is connected and $G \cup H$ is not connected. Which one of the following statements is true for all such G and H?* ***(MCQ)***

(a) If $G \cap H = \phi$, then H is connected

(b) If $G \cap H = \phi$, then H is not connected

(c) If $G \cap H \neq \phi$, then H is connected

(d) If $G \cap H \neq \phi$, then H is not connected

47. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \int_{-5}^{x} (t-1)^3 \, dt$. *In which of the following interval(s), f takes the value 1*

(MSQ)

(a) [-6, 0] (b) [-2, 4] (c) [2, 8] (d) [6, 12]

48. *Which of the following condition(s) implies (imply) the convergence of a sequence $\{x_n\}$ of real numbers*

(MSQ)

(a) Given $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $|x_{n+1} - x_n| < \varepsilon$

(b) Given $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\frac{1}{(n+1)^2} | x_{n+1} - x_n | < \varepsilon$

(c) Given $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $(n+1)^2 |x_{n+1} - x_n| < \varepsilon$

(d) Given $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all m, n with $m > n \geq n_0$, $|x_m - x_n| < \varepsilon$

49. *Which of the following statements is (are) true on the interval* $\left(0, \frac{\pi}{2}\right)$? **(MSQ)**

(a) $\cos x < \cos(\sin x)$

(b) $\tan x < x$

(c) $\sqrt{1+x} < 1 + \frac{x}{2} - \frac{x^2}{8}$

(d) $\frac{1-x^2}{2} < \ell n(2+x)$

50. *Let f,g : [0,1]⟶[0,1] be functions. Let R(f) and R(g) be the ranges of f and g, respectively. Which of the following statements is (are) true?* ***(MSQ)***

(a) If $f(x) \leq g(x)$ for all $x \in [0, 1]$, then $\sup R(f) \leq \inf R(g)$

(b) If $f(x) \leq g(x)$ for some $x \in [0, 1]$, then $\inf R(f) \leq \sup R(g)$

(c) If $f(x) \leq g(y)$ for some $x, y \in [0, 1]$, then $\inf R(f) \leq \sup R(g)$

(d) If $f(x) \leq g(y)$ for all $x, y \in [0, 1]$, then $\sup R(f) \leq \inf R(g)$

51. *If the power series* $\sum_{n=0}^{\infty} \frac{n!}{n^n} x^{2n}$ *converges for* $|x| < c$ *and diverges for* $|x| > c$*, then the value of c, correct upto three decimal places, is* _________ *(NAT)*

52. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) = \begin{cases} x^6 - 1, & x \in \mathbb{Q} \\ 1 - x^6, & x \notin \mathbb{Q} \end{cases}$ **(NAT)**

The number of points at which f is continuous, is ________

53. *Let $f : (0, 1) \to \mathbb{R}$ be a continuously differentiable function such that f' has finitely many zeros in (0, 1) and f' changes sign at exactly two of these points. Then for any $y \in \mathbb{R}$, the maximum number of solutions to $f(x) = y$ in (0, 1) is ________* **(NAT)**

54. *The limit* $\lim_{n \to \infty} \sum_{k=2}^{n} \frac{1}{k^3 - k}$ *is equal to* ________ **(NAT)**

55. *The coefficient of* $\left(x-\frac{\pi}{4}\right)^3$ *in the Taylor series expansion of the function* $f(x)=3\sin x\cos\left(x+\frac{\pi}{4}\right)$, $x\in\mathbb{R}$ *about the point* $\frac{\pi}{4}$, *correct upto three decimal places, is* ______ ***(NAT)***

56. *If* $\int_0^x (e^{-t^2} + \cos t)\, dt$ *has the power series expansion* $\sum_{n=1}^{\infty} a_n x^n$, *then* a_5, *correct upto three decimal places, is equal to* ________ ***(NAT)***

57. *The limit* $\lim_{x \to 0^+} \frac{9}{x}\left(\frac{1}{\tan^{-1} x} - \frac{1}{x}\right)$ *is equal to* _________ ***(NAT)***

58. *The sequence* $\{s_n\}$ *of real numbers given by* $s_n = \frac{\sin\frac{\pi}{2}}{1.2} + \frac{\sin\frac{\pi}{2^2}}{2.3} + \ldots + \frac{\sin\frac{\pi}{2^n}}{n.(n+1)}$ *is* **(MCQ)**

(a) a divergent sequence *(b) an oscillatory sequence*

(c) not a Cauchy sequence *(d) a Cauchy sequence*

59. *Let $f : [-1,1] \to \mathbb{R}$ be a continuous function. Then the integral* $\int_0^{\pi} x f(\sin x)\,dx$ *is equivalent to* ***(MCQ)***

(a) $\frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$ *(b)* $\frac{\pi}{2}\int_0^{\pi} f(\cos x)\,dx$ *(c)* $\pi\int_0^{\pi} f(\cos x)\,dx$ *(d)* $\pi\int_0^{\pi} f(\sin x)\,dx$

60. *The value of* $\lim_{(x,y)\to(2,-2)} \frac{\sqrt{(x-y)}-2}{x-y-4}$ *is* ***(MCQ)***

(a) 0 *(b)* $\frac{1}{4}$ *(c)* $\frac{1}{3}$ *(d)* $\frac{1}{2}$

61. *Let S be a closed subset of* $\mathbb{R}$, *T a compact subset of* $\mathbb{R}$ *such that* $S \cap T \neq \phi$. *Then,* $S \cap T$ *is* **(MCQ)**

(a) closed but not compact
(b) not closed
(c) compact
(d) neither closed nor compact

62. *Let S be the series* $\sum_{k=1}^{\infty} \frac{1}{(2k-1)2^{(2k-1)}}$ *and T be the series* $\sum_{k=2}^{\infty} \left(\frac{3k-4}{3k+2}\right)^{\frac{(k+1)}{3}}$ *of real numbers. Then, which one of the following is TRUE?* ***(MCQ)***

(a) Both the series S and T are convergent

(b) S is convergent and T is divergent

(c) S is divergent and T is convergent

(d) Both the series S and T are divergent

63. Let $\{a_n\}$ be a sequence of positive real numbers satisfying $\frac{4}{a_{n+1}} = \frac{3}{a_n} + \frac{a_n^3}{81}$, $n \geq 1$, $a_1 = 1$. Then, all the terms of the sequence lie in

(MCQ)

(a) $\left[\frac{1}{2}, \frac{3}{2}\right]$ (b) $[0, 1]$ (c) $[1, 2]$ (d) $[1, 3]$

75. *Let $\{s_n\}$ be a sequence of real numbers given by* $s_n = 2^{(-1)^n}\left(1-\frac{1}{n}\right)\sin\frac{n\pi}{2}$, $n \in \mathbb{N}$.

Then the least upper bound of the sequence $\{s_n\}$ is ______________ ***(NAT)***

76. Let $\{s_k\}$ be a sequence of real numbers, where $s_k = k^{\alpha/k}$, $k \geq 1$, $\alpha > 0$.
Then, $\lim_{n\to\infty}(s_1 s_2 \ldots s_n)^{1/n}$ is ____________

(NAT)

77. *If* $f : (-1, \infty) \to \mathbb{R}$ *defined by* $f(x) = \dfrac{x}{1+x}$ *is expressed as* $f(x) = \dfrac{2}{3} + \dfrac{1}{9}(x-2) + \dfrac{c(x-2)^2}{(1+\xi)^3}$,
where ξ *lies between 2 and x, then, the value of c is* ________________ ***(NAT)***

78. *The radius of convergence of the power series* $\sum_{n=1}^{\infty} \frac{(-4)^n}{n(n+1)}(x+2)^{2n}$ *is* ___________ ***(NAT)***

79. *Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function such that* $\int_0^x f(t)dt = -2 + \frac{x^2}{2} + 4x\sin 2x + 2\cos 2x.$

Then, the value of $\frac{1}{\pi} f\left(\frac{\pi}{4}\right)$ *is* __________

(NAT)

80. *The value of* $\lim_{n \to \infty} \left(8n - \frac{1}{n}\right)^{\frac{(-1)^n}{n^2}}$ *is equal to* ________ ***(NAT)***

81. *Let $f_1(x)$, $f_2(x)$, $g_1(x)$, $g_2(x)$ be differentiable functions on $\mathbb{R}$. Let* $F(x)=\begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix}$ *be the determinant of the matrix* $\begin{bmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{bmatrix}$. *Then $F'(x)$ is equal to* **(MCQ)**

(a) $\begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1'(x) \\ f_2'(x) & g_2(x) \end{vmatrix}$

(b) $\begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1'(x) \\ f_2(x) & g_2'(x) \end{vmatrix}$

(c) $\begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} - \begin{vmatrix} f_1(x) & g_1'(x) \\ f_2(x) & g_2'(x) \end{vmatrix}$

(d) $\begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}$

83. $\lim_{n \to \infty} \frac{\pi}{n} \sum_{k=1}^{n} \sin\left(\frac{\pi}{2} + \frac{5\pi}{2} \cdot \frac{k}{n}\right) =$ (MCQ)

(a) $\frac{2\pi}{5}$ (b) $\frac{5}{2}$ (c) $\frac{2}{5}$ (d) $\frac{5\pi}{2}$

84. If $\lim_{T\to\infty}\int_0^T e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$, then $\lim_{T\to\infty}\int_0^T x^2 e^{-x^2}dx =$ (MCQ)

(a) $\frac{\sqrt{\pi}}{4}$ (b) $\frac{\sqrt{\pi}}{2}$ (c) $\sqrt{2\pi}$ (d) $2\sqrt{\pi}$

85. *Let S be an infinite subset of* $\mathbb{R}$ *such that* $S \setminus \{\alpha\}$ *is compact for some* $\alpha \in S$. *Then which one of the following is TRUE?* ***(MCQ)***

(a) *S is a connected set*

(b) *S contains no limit points*

(c) *S is a union of open intervals*

(d) *Every sequence in S has a subsequence converging to an element in S*

86. *Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(2) = 2$ and $|f(x) - f(y)| \leq 5(|x - y|)^{3/2}$ for all $x \in \mathbb{R}, y \in \mathbb{R}$. Let $g(x) = x^3 f(x)$. Then $g'(2) =$* **(MCQ)**

(a) 5 (b) $\frac{15}{2}$ (c) 12 (d) 24

87. $\sum_{n=1}^{\infty} \tan^{-1} \frac{2}{n^2} =$ ***(MCQ)***

(a) $\frac{\pi}{4}$ *(b)* $\frac{\pi}{2}$ *(c)* $\frac{3\pi}{4}$ *(d)* π

88. *Let* $f : \mathbb{R} \to [0, \infty)$ *be a continuous function. Then which one of the following is NOT TRUE?* **(MCQ)**

(a) *There exists* $x \in \mathbb{R}$ *such that* $f(x) = \dfrac{f(0) + f(1)}{2}$

(b) *There exists* $x \in \mathbb{R}$ *such that* $f(x) = \sqrt{f(-1)\, f(1)}$

(c) *There exists* $x \in \mathbb{R}$ *such that* $f(x) = \int_{-1}^{1} f(t)\, dt$

(d) *There exists* $x \in \mathbb{R}$ *such that* $f(x) = \int_{0}^{1} f(t)\, dt$

89. *Let* $f(x, y) = \dfrac{x^2 y}{x^2 + y^2}$ *for* $(x, y) \neq (0, 0)$. *Then*

(a) $\dfrac{\partial f}{\partial x}$ *and* f *are bounded*

(b) $\dfrac{\partial f}{\partial x}$ *is bounded and* f *is unbounded*

(c) $\dfrac{\partial f}{\partial x}$ *is unbounded and* f *is bounded*

(d) $\dfrac{\partial f}{\partial x}$ *and* f *are unbounded*

90. Let $0 < a_1 < b_1$. For $n \geq 1$, define $a_{n+1} = \sqrt{a_n b_n}$ and $b_{n+1} = \frac{a_n + b_n}{2}$.

Then which one of the followings is NOT TRUE?

(a) Both $\{a_n\}$ and $\{b_n\}$ converge, but the limits are not equal

(b) Both $\{a_n\}$ and $\{b_n\}$ converge and the limits are equal

(c) $\{b_n\}$ is a decreasing sequence

(d) $\{a_n\}$ is an increasing sequence

91. $\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{3}+\sqrt{6}}+\frac{1}{\sqrt{6}+\sqrt{9}}+\ldots+\frac{1}{\sqrt{3n}+\sqrt{3n+3}}\right)=$ ***(MCQ)***

(a) $1+\sqrt{3}$ *(b)* $\sqrt{3}$ *(c)* $\frac{1}{\sqrt{3}}$ *(d)* $\frac{1}{1+\sqrt{3}}$

92. *The interval of convergence of the power series* $\sum_{n=1}^{\infty} \frac{1}{(-3)^{n+2}} \frac{(4x-12)^n}{n^2+1}$ *is* **(MCQ)**

(a) $\frac{10}{4} \le x < \frac{14}{4}$ (b) $\frac{9}{4} \le x < \frac{15}{4}$ (c) $\frac{10}{4} \le x \le \frac{14}{4}$ (d) $\frac{9}{4} \le x \le \frac{15}{4}$

96. *Let $\{x_n\}$ be a real sequence such that $7x_{n+1} = x_n^3 + 6$ for $n \geq 1$. Then which of the following statements are TRUE?* ***(MSQ)***

(a) If $x_1 = \frac{1}{2}$, then $\{x_n\}$ converges to 1

(b) If $x_1 = \frac{1}{2}$, then $\{x_n\}$ converges to 2

(c) If $x_1 = \frac{3}{2}$, then $\{x_n\}$ converges to 1

(d) If $x_1 = \frac{3}{2}$, then $\{x_n\}$ converges to -3

97. $\frac{1}{2\pi}\left(\frac{\pi^3}{1!\,3}-\frac{\pi^5}{3!\,5}+\frac{\pi^7}{5!\,7}-\ldots+\frac{(-1)^{n-1}\pi^{2n+1}}{(2n-1)!\,(2n+1)}+\ldots\right)=$ ________

98. $\left(\int_0^1 x^4(1-x)^5\,dx\right)^{-1} =$ ________

99. *For x > 0, let [x] denote the greatest integer less than or equal to x.*

Then $\lim_{x\to 0^+} x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+\ldots+\left[\frac{10}{x}\right]\right) =$ ____________

100. *If* $y(x)=\int_{\sqrt{x}}^{x}\frac{e^t}{t}\,dt, x>0$, *then* $y'(1)=$ ________