Method of Least Squares:-

Consider a variable Y [random], & it depends on another variable X [non-random].

Deterministic part

Relationship b/w X & Y: $Y = \alpha + \beta X + \epsilon$ → error.

∴ R.S: $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$.

using this random sample estimate α, β.

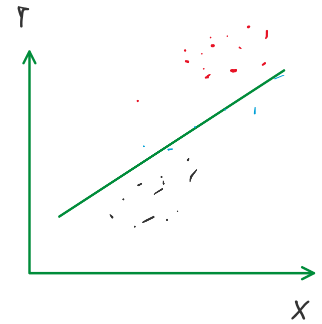
Let $\hat{\alpha}, \hat{\beta}$ be the estimates of α, β [unknown popln parameter]

∴ Method of Least squares:

Estimated model: $\hat{Y}_i = \hat{\alpha} + \hat{\beta} X_i$, $i = 1, 2, \ldots, n$.

Actual model: $Y_i = \alpha + \beta X_i + \epsilon_i$, $i = 1, 2, \ldots, n$.

$\min\limits_{\hat{\alpha}, \hat{\beta}} \sum_{i=1}^{n} |\epsilon_i|$ / $\min\limits_{\hat{\alpha}, \hat{\beta}} \sum_{i=1}^{n} \epsilon_i^2$ → Algebric convenience.



OLS: $\min\limits_{\hat{\alpha}, \hat{\beta}} \sum_{i=1}^{n} \epsilon_i^2$; ϵ_i = error in estimation

$$\therefore \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta} X_i)^2$$

$\sum_{i=1}^{n} (x_i - \bar{x}) = 0$

$\frac{1}{n} \sum |x_i - \bar{x}| = MD_{\bar{x}}$

$\frac{1}{n} \sum (x_i - \bar{x})^2 = Var(x)$

∴ For Minimization:-

$$\frac{\partial \sum \epsilon_i^2}{\partial \hat{\alpha}} = 0 \Rightarrow -\sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta} X_i)(2) = 0$$

$$\sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta} X_i) = 0 \quad \text{---- (i)}.$$

$i=1$

$$\frac{\partial \sum \epsilon_i^2}{\partial \hat\beta} = 0 \Rightarrow (-2)\sum_{i=1}^{n}(Y_i - \hat\alpha - \hat\beta X_i)X_i = 0$$

$$\sum_{i=1}^{n}(Y_i - \hat\alpha - \hat\beta X_i)X_i = 0 \quad \text{--- (ii)}$$

solve for $\hat\alpha, \hat\beta$.

$$a_1 x + b_1 y = c_1.$$
$$a_2 x + b_2 y = c_2$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$(i) \Rightarrow \sum_{i=1}^{n}(Y_i - \hat\alpha - \hat\beta X_i) = 0.$$

$$\sum Y_i - \sum\hat\alpha - \hat\beta\sum X_i = 0.$$

$$\frac{n\hat\alpha}{n} + \frac{\hat\beta\sum X_i}{n} = \frac{\sum Y_i}{n}$$

$$\hat\alpha + \hat\beta\bar X = \bar Y \quad \text{---- (ia)}$$

$$(ii) \Rightarrow \sum_{i=1}^{n}(Y_i - \hat\alpha - \hat\beta X_i)X_i = 0$$

$$\sum Y_i X_i - \hat\alpha\sum X_i - \hat\beta\sum X_i^2 = 0$$

$$\hat\alpha\left(\sum X_i\right) + \hat\beta\left(\sum X_i^2\right) = \sum Y_i X_i \quad \text{--- (iia)}$$

Summarizing:
$$\hat\alpha + (\bar X)\cdot\hat\beta = \bar Y \quad \text{--- (ia)}$$
$$\left(\sum X_i\right)\hat\alpha + \left(\sum X_i^2\right)\hat\beta = \left(\sum X_i Y_i\right) \quad \text{--- (iia)}$$

$$\hat\beta = \frac{\begin{vmatrix} 1 & \bar Y \\ \sum X_i & \sum X_i Y_i \end{vmatrix}}{\begin{vmatrix} 1 & \bar X \\ \sum X_i & \sum X_i^2 \end{vmatrix}} = \frac{\left(\sum X_i Y_i - \bar Y\sum X_i\right)/n}{\left(\sum X_i^2 - \bar X\sum X_i\right)/n}$$

$$= \frac{\frac{1}{n}\sum X_i Y_i - \bar X\bar Y}{\frac{1}{n}\sum X_i^2 - \bar X^2}.$$

$$= \frac{\overbrace{\cancel{\tfrac{1}{n}}\sum(X_i - \bar X)(Y_i - \bar Y)}^{=\,cov(X,Y)}}{\underbrace{\cancel{\tfrac{1}{n}}\sum(X_i - \bar X)^2}_{=\,Var(X)}} = \frac{\sum(X_i - \bar X)(Y_i - \bar Y)}{\sum(X_i - \bar X)^2}$$

$\frac{1}{n}\sum(x_i-\bar{x})^2$ $\qquad \sum(x_i-x)^2$

$\approx Var(X)$

From (ia): $\hat{\alpha} + \bar{x}\cdot\hat{\beta} = \bar{Y}$

$\Rightarrow \hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$ [Replacing value of $\hat{\beta}$].

(Elimination)
Summarizing: $1\,\hat{\alpha} + (\bar{X})\cdot\hat{\beta} = \bar{Y}$ --- (ia) $\times (\sum x_i)$

$(\sum x_i)\hat{\alpha} + (\sum x_i^2)\hat{\beta} = (\sum x_i Y_i)$ --- (iia) $\times 1$.

Subtract :- $\frac{\bar{X}\cdot(\sum x_i)}{n}\hat{\beta} - \frac{(\sum x_i^2)}{n}\hat{\beta} = \frac{(\bar{Y}\sum x_i)}{n} - \frac{\sum x_i Y_i}{n}$

$$\left[\bar{X}^2 - \frac{1}{n}\sum x_i^2\right]\hat{\beta} = \bar{X}\bar{Y} - \frac{1}{n}\sum x_i Y_i$$

$$\left[\frac{1}{n}\sum x_i^2 - \bar{X}^2\right]\hat{\beta} = \frac{1}{n}\sum x_i Y_i - \bar{X}\bar{Y}$$

$$\hat{\beta} = \frac{\frac{1}{n}\sum x_i Y_i - \bar{X}\bar{Y}}{\frac{1}{n}\sum x_i^2 - \bar{X}^2}.$$

HW.

Note: $Y = \alpha + \beta X + \gamma X^2$ ......

R.S $(X_1, Y_1), \ldots, (X_n, Y_n) \Rightarrow$ use OLS to estimate this model?

Use OLS to find $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$