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Normal Distribution → distribution of a continuous random variable (pdf).

parameters

(population mean μ

and " s.d σ)

X is a cont r.v with μ and σ follow normal distribution with ~~pdf~~

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}} \quad ; -\infty < x < \infty$$

① condition for p.d.f → a) $f(x) \geq 0$ for all values x

b) $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Proof

N.D is a pdf.

(i) for all values of x, $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

let $\boxed{t = \frac{x-\mu}{\sigma}}$

$dt = \frac{dx}{\sigma}$

$\Rightarrow \boxed{dx = \sigma \cdot dt}$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} \, \cancel{\sigma} \cdot dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} \, dt$$

↳ even function

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t^2/2} \, dt \qquad \left[e^{-t^2/2} \text{ is an even fn} \right]$$

Again let $y = t^2/2$

$dy = \frac{\cancel{2}t \, dt}{\cancel{2}}$ $\Rightarrow dt = \frac{dy}{t} = \frac{dy}{\sqrt{2y}}$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-y} \frac{dy}{\sqrt{2y}}$$

$\frac{1}{\sqrt{y}} = y^{-1/2}$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{2}} e^{-y} \cdot y^{-1/2} \cdot dy$$

$$= \frac{2}{2\sqrt{\pi}} \left(\int_{0}^{\infty} e^{-y} \cdot y^{-1/2} \, dy \right)$$

$$= \frac{\cancel{2}}{\cancel{2}\sqrt{\pi}} \left(\int_0^{\infty} e^{-y} \cdot y^{-(1/2)} \, dy \right) \quad \Gamma$$

$$= \frac{1}{\sqrt{\pi}} \Gamma(1/2)$$

$$\boxed{\Gamma(1/2) = \sqrt{\pi}}$$

$$= \frac{1}{\cancel{\sqrt{\pi}}} \times \cancel{\sqrt{\pi}} = 1 \text{ (proved)}$$

(2) Mean of Normal Distribution

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \, dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $\frac{x-\mu}{\sigma} = t$

then $\frac{dx}{\sigma} = dt$

$\Rightarrow dx = \sigma \cdot dt$

$$= \int_{-\infty}^{\infty} (\mu + \sigma t) \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \cancel{\sigma} \cdot dt$$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} dt + \frac{\sigma}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} t \cdot e^{-t^2/2} dt}_{0}$$

$\because e^{-t^2/2}$ is even and $t e^{-t^2/2}$ is odd

$\int f(x) = 0$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} dt$$

$$= 2\mu \int_0^{\infty} e^{-t^2/2} dt$$

$$= \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} e^{-t^2/2}\, dt$$

$$\text{let } y = \frac{t^2}{2} \quad \therefore dy = \frac{\cancel{2}t\,dt}{\cancel{2}}$$

$$= \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} e^{-y^2} \frac{dy}{\sqrt{2y}}$$

$$= \frac{1\mu}{\sqrt{\pi}} \int_0^{\infty} e^{-y^2}\, \underbrace{y^{-1/2}}\, dy$$

$$\Gamma_{1/2} = \sqrt{\pi}$$

$$\frac{1}{2}-1 \qquad \alpha - 1$$

$$\boxed{E(x) = \frac{\mu}{\cancel{\sqrt{\pi}}} \times \cancel{\sqrt{\pi}} = \mu = \text{mean}}$$

variance of x

$$Var(x) = E(x^2) - E(x)^2$$

Raw moments → deviation from any arbitrary constant (A)

$$\mu_r' = \frac{1}{n} \Sigma (x_i - A)^r$$

(rth order raw moment)

$r = 1 \Rightarrow \mu_1' = \frac{1}{n} \Sigma (x_i - A)$

$r = 2 \Rightarrow \mu_2' = \frac{1}{n} \Sigma (x_i - A)^2$

⋮

Central moment → - - - . from mean ($\bar{x}$)

$$\mu_r = \frac{1}{n} \Sigma (x_i - \bar{x})^r$$

(rth order central moment)

$r = 1 \Rightarrow \mu_1 = \frac{1}{n} \Sigma (x_i - \bar{x}) = 0$

$r = 2 \Rightarrow \mu_2 = \frac{1}{n} \Sigma (x_i - \bar{x})^2 = \sigma^2$

$r = 2 \Rightarrow \mu_2 = \frac{1}{n} \Sigma (x_i - \bar{x})^2 = \sigma^2$

$r = 3 \Rightarrow \mu_3 = \frac{1}{n} \Sigma (x_i - \bar{x})^3 \Rightarrow$ skewness

Properties

(1) $E(x) = \mu$ (2) N.D $\Rightarrow$ Mean = Med = Mode = μ.

(3) ND is symmetrical $\Rightarrow$ No skewness (=0)

odd-order central moment is 0.

$\bar{x} = \frac{1}{n} \Sigma x_i$

$E(x)$

$\mu_r = \frac{1}{n} \Sigma (x - \bar{x})^r$

E

$$\mu_{2r+1} = \frac{1}{n} \Sigma (x - \mu)^{2r+1}$$

$$\mu_{2r+1} = E(x - \mu)^{2r+1}$$ $\Rightarrow$ let us define an odd-order central moment.

$$= \int_{-\infty}^{\infty} (x-\mu)^{2r+1} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2r+1} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $t = x - \mu / \sigma$

$dt = \frac{dx}{\ }$

$$dt = \frac{dx}{\sigma}$$

$$= \int_{-\infty}^{\infty} (\sigma t)^{2r+1} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-t^2/2} \cancel{\sigma}\, dt$$

$$= \frac{\sigma^{2r+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^{2r+1} \cdot e^{-t^2/2}\, dt$$

$$\mu_{2r+1} = 0 \quad \text{(proved)}$$

$\therefore \mu_1 = 0$, $\mu_3 = 0$, $\mu_5 = 0$ . . . .

Let the even order central moment be

$$\mu_{2r} = E(X-\mu)^{2r} = \int_{-\infty}^{\infty} (x-\mu)^{2r} f(x)\, dx$$

$$\mu_{2r} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^{2r} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$2r$

a b

$-\infty$ 0 ∞

$2(0 \to \infty)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (t \cdot \sigma)^{2r} e^{-\frac{1}{2}t^2} \cdot dt$$

$$= \frac{\sigma^{2r}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{t^{2r} \cdot e^{-\frac{1}{2}t^2}}_{\text{even fn}} dt$$

$$= \frac{\sigma^{2r}}{} \int^{\infty} 2 \cdot t^{2r} e^{-t^2/2}\, dt$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_0 2 \cdot t^{2r} e^{-t^2/2}\, dt$$

let $y = \frac{t^2}{2}$

$dy = \frac{\cancel{2}t}{\cancel{2}}\, dt$

$\frac{dy}{t} = dt$

$dt = \frac{dy}{\sqrt{2y}}$

$$= \frac{2\sigma^{2r}}{\sqrt{2\pi}} \int_0^\infty (2y)^r e^{-y} \cdot \frac{dy}{\sqrt{2y}}$$

$$= \frac{\sigma^{2r}}{\sqrt{\pi}} \int_0^\infty 2^r \cdot y^r\, e^{-y}\, y^{-1/2}\, dy$$

$$= \frac{\sigma^{2r} \cdot 2^r}{\sqrt{\pi}} \int_0^\infty e^{-y}\, y^{r-1/2}\, dy$$

$$= \frac{\sigma^{2r} \cdot 2^r}{\sqrt{\pi}} \int_0^\infty e^{-y}\, y^{\left(r+\frac{1}{2}\right)-1}\, dy$$

$y^{-1/2}$

$y^{(1/2)-1} = \Gamma 1/2$

$$\boxed{\int_0^\infty e^{-y}\, y^{\alpha-1}\, dy = \Gamma(\alpha)}$$

$\Gamma(5) = 5\,\Gamma(4)$

$= 5 \times 4\,\Gamma 3$

$= 5 \times 4 \times 3\,\Gamma 2$

$= 5 \times 4 \times 3\,\Gamma 1$

$= 5 \times 4 \times 3 \times 2 \times 1/2\, \Gamma 1/2 \ldots$

$\Gamma(n) = (n-1)\,\Gamma n-1$

$(n-2)\,n-3\,\Gamma n-3$

$$= \frac{2^r \cdot \sigma^{2r}}{\sqrt{\pi}}\, \Gamma\left(r+\frac{1}{2}\right)$$

$$= \frac{2^r \cdot \sigma^{2r}}{\sqrt{\pi}} \left[\left(r-\tfrac{1}{2}\right)\left(r-\tfrac{3}{2}\right)\left(r-\tfrac{5}{2}\right) \cdots \tfrac{3}{2} \cdot \tfrac{1}{2}\,\Gamma\tfrac{1}{2}\right]$$

r times.

$$= \frac{2^r \cdot \sigma^{2r}}{\sqrt{\pi}} \underbrace{\left(\frac{2r-1}{2}\right)\left(\frac{2r-3}{2}\right)\left(\frac{2r-5}{2}\right) \cdots \frac{1}{2} \times \sqrt{\pi}}_{}$$

$$= \frac{\cancel{2^r} \cdot \sigma^{2r}}{\cancel{\sqrt{\pi}}} \cdot \frac{(2r-1)(2r-3) \cdots \cancel{\sqrt{\pi}}}{\cancel{2^r}}$$

$$\boxed{\mu_{2r} = \sigma^{2r}(2r-1)(2r-3)\cdots 3\cdot 1.}$$

Now, varian $(x) = \mu_2$

$\therefore$ if $r=1 \Rightarrow \mu_2 = \sigma^{2\times 1}(2\times 1 - 1)$

$$\boxed{\mu_2 = \sigma^2 \times 1 = \sigma^2 \ (\text{var}(x))}$$

if $r=2 \Rightarrow \mu_4 = \sigma^4 \cdot 3 \cdot 1. = 3\cdot\sigma^4.$

Moment - Generating Function

$$m_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - t\mu + t\mu - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{e^{t\mu}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{+t(x-\mu) - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{e^{\cdot\cdot}}{\sigma\sqrt{2\pi}} \int_{-\infty} e^{\cdots 2\sigma^2} dx$$

$$= \frac{e^{t\mu}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left\{(x-\mu)^2 - 2\sigma^2 t(x-\mu)\right\}} dx$$

$$= \frac{e^{t\mu}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left\{(x-\mu)^2 - 2(x-\mu)\sigma^2 t + \sigma^4 t^2 - \sigma^4 t^2\right\}} dx$$

$$= \frac{e^{\mu t + \frac{\sigma^2 t}{2}}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x - \underbrace{\mu - t\sigma^2}_{\text{mean}})^2} dx$$

the integrant is the p.d.f of a normal dis with mean $\mu + t\sigma^2$ and s.d σ

$\therefore \forall x \; f(x) \geq 0$

an $\int f(x)\,dx = 1$

$$= e^{\mu t + \frac{\sigma^2 t}{2}}$$

Mgf of N.D.

(Proved).