30 September 2023 19:04

Continuous Random Variables

Probability Distributions

Random variables → Discrete R.V (a variable can have only distinct set of outcomes)
↳ Continuous R.V. (a variable can take any possible value within a given range).

Prob Distribution of a Discrete R.V ⇒ pmf [Prob mass fn].

Eg: $X \sim f(x)$ ⇒ valid pmf: $P(X=x) \geq 0 \ \forall x$ & $\sum_x P(X=x) = 1$.

Prob Distribution of a continuous R.V ⇒ pdf [Prob density fn].

Eg: $X \sim f(x)$ ⇒ valid pdf: $f(x) \geq 0 \ \forall x$ & $\int_{-\infty}^{+\infty} f(x) = 1$.

Eg: Suppose $f(x) = \begin{cases} \frac{1}{\sqrt{x(1-x)}}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$. Check if $f(x)$ is a valid pdf.

$0 < x < 1 \Rightarrow f(x) = \frac{1}{\sqrt{\underbrace{x}_{>0}\underbrace{(1-x)}_{>0}}} \geq 0$.

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}} dx$$

Let $x = \sin^2\theta$. $\Rightarrow \theta = \sin^{-1}\sqrt{x}$.

$dx = \sin 2\theta \, d\theta$ $\quad x = 0, \ \theta = 0$.

$x = 1, \ \theta = \pi/2$

$$\int_0^{\pi/2} \frac{1}{\sqrt{\sin^2\theta \cdot \cos^2\theta}} \sin 2\theta \, d\theta = \int_0^{\pi/2} \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2\left[\frac{\pi}{2}\right] = \pi \neq 1$$

∴ $f(x)$ is not a valid pdf.

Note: $\int_0^1 \frac{1}{\sqrt{x(1-x)}} dx = \pi$.

Note: $\int_0^1 \frac{1}{\sqrt{x(1-x)}} \cdot dx = \pi$

$\frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{x(1-x)}} dx = 1 \Rightarrow \int_0^1 \frac{1}{\pi\sqrt{x(1-x)}} dx = 1$.

normalizing constant. This is a valid pdf.

Eg: $f(x) = c \cdot \frac{1}{\sqrt{x(1-x)}}, \; 0 < x < 1$

Cdf of a Continuous Random variable:-

(Cumulative Density function)

Let $X \sim f(x)$.

$\therefore$ cdf of r.v X is defined as $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\, dx$.

Note: $\boxed{\frac{dF(x)}{dx} = f(x)}$ and $\left.\begin{array}{l} F(-\infty) = 0 \\ F(\infty) = 1 \end{array}\right\}$

(*) Expectation of a r.v $E(X) = \int_{-\infty}^{+\infty} x f(x)\, dx = \mu$.

(*) Variance of a r.v $Var(X) = E(X^2) - [E(X)]^2$

where $E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\, dx$.

(*) Median of the prob distribution: $\int_{-\infty}^{\tilde{\mu}} f(x)\, dx = 0.5$

Let median $= \tilde{\mu}$.

(*) Mode of the prob distribution: $\frac{df(x)}{dx} = 0 \Rightarrow$ solves for mode

[Assume $f(x)$ is differentiable].

(*) Skewness of the prob distribution:

(x) Skewness of the prob distribution:

γ_1 and β_1 measures.

Defn: $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} \Rightarrow \gamma_1 = \sqrt{\beta_1}$.

μ_3 : 3rd order central moment $= E(X-\mu)^3$.

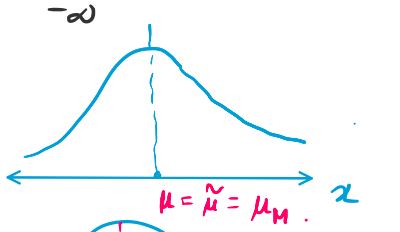
μ_2 : 2nd order central moment $= E(X-\mu)^2 = Var(X)$

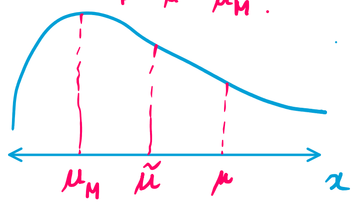
$$E(X-\mu)^3 = \int_{-\infty}^{+\infty} (x-\mu)^3 f(x)\, dx \qquad E(X-\mu)^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\, dx$$

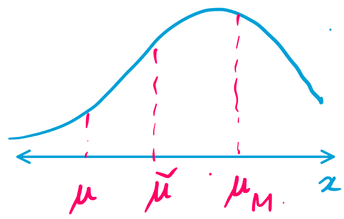
Note: If $\gamma_1 = 0 \Rightarrow$ distn is symmetric.

If $\gamma_1 > 0 \Rightarrow$ distn is positively skewed

If $\gamma_1 < 0 \Rightarrow$ distn is negatively skewed







(x) If distn is symm: Mean = Median = Mode

(x) If distn is positively skewed:

Mean > Median > Mode.

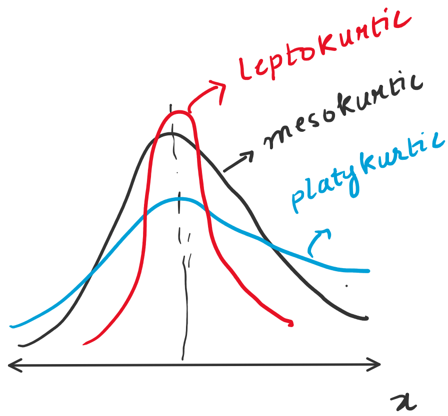
(x) If distn is negatively skewed:

Mean < Median < Mode.

(x) Kurtosis of the probability distribution:

(how peaked is the distribution)

We use β_2 and γ_2 measures.



$$\beta_2 = \frac{\mu_4}{\mu_2^2} \quad \text{and} \quad \boxed{\gamma_2 = \beta_2 - 3}.$$

where: μ_4 = 4th order central moment $= E(X-\mu)^4$.

$$= \int_{-\infty}^{+\infty} (x-\mu)^4 f(x)\, dx.$$

(x) If $\gamma_2 > 0 \Rightarrow$ distn is leptokurtic.

If $\gamma_2 = 0 \Rightarrow$ distn is mesokurtic

(*) If $\gamma_2 > 0 \Rightarrow$ distn is leptokurtic.

(*) If $\gamma_2 = 0 \Rightarrow$ distn is mesokurtic

(*) If $\gamma_2 < 0 \Rightarrow$ distn is platykurtic