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Differential Equations

1st order

- Homogenous: $y'(t) + b\,y(t) = 0$
  - First derivative
- Non-homogenious: $y'(t) + b(y(t)) = c$
  - non-zero const

2nd order

- Homomogenious: $y''(t) + a\,y'(t) + y(t) = 0$
- Non-homogenous: $y''(t) + a\,y'(t) + y(t) = c$

1st order homogenious differential equation:

$$\boxed{\frac{dy}{dt} + g\,y = 0}$$

$$\boxed{y(t) = Ae^{-\int g\,dt}}$$

also called the complementary function

Q. Obtain the general solution of $\frac{dy}{dt} + 2y(t) = 0$. the definite solution given $y(0) = 1$.

$$\frac{dy}{dt} + 2y = 0$$

$$\therefore y = -Ae^{-\int 2\,dt}$$

$$\therefore y = Ae^{-2t+c}$$

$$y = A e^{-2t} \cdot e^{c}$$

$$y = \lambda e^{-2t}$$

$$[\lambda = Ae^{c}]$$

Given at $t=0$, $y(0) = 1$

$$\lambda e^{-2(0)} = 1$$

$$\lambda = 1$$

$\therefore$ Required solution is $y(t) = 1 \cdot e^{-2t}$

or $y(t) = e^{-2t}$ (ans).

2) Solve the equation $3 \cdot \frac{dy}{dt} + 6y = 0$. The initial condition given is $y(0) = 2$.

$$3 \cdot \frac{dy}{dt} + 6y = 0$$

$$\frac{dy}{dt} + 2y = 0$$

$\int 2\,dt$

$$\frac{dy}{dt} + 2y = 0$$

$$y(t) = e^{-\int 2\,dt} \cdot A$$

$$= e^{-2t+c} \cdot A$$

$$y(t) = e^{-2t} \cdot \lambda$$

At initial $t = 0$ then $y(0) = 2$

$e^{-2(0)} \lambda = 2$

$\lambda = 2$

$\therefore$ $y(t) = 2 \cdot e^{-2t}$ (ans).

② 1st order Non-homogeneous Differential equation.

$$\frac{dy}{dt} + ay = b$$

$$\Rightarrow \quad y(t) = PI + CF$$

$$y_p + y_c$$

intemporal-equilibrium value.

$$y = e^{-\int a\,dt}\left(A + \int b e^{\int a\,dt}\,dt\right)$$

$$= e^{-at}\left(A + \int b e^{at} \cdot dt\right)$$

$$= e^{-at}\left(A + \frac{b e^{at}}{a}\right)$$

$$y(t) = \underbrace{A e^{-at}}_{CF} + \underbrace{\frac{b}{a}}_{PI}$$

How to calculate PI (another method)

~~y doesnot~~ change with t ie $\frac{dy}{dt} = 0$

PI

let $y(t) = c$ (some const)

ie $dy/dt = 0$

$\therefore \frac{dy}{dt} + ay = b$

$0 + a \cdot c = b$

$\boxed{c = b/a}$ PI

Q $\frac{dy}{dt} + 5y = 15$ given $y_0 = 8$ — (1)

1st order, homogeneous, differential equation.

PI : let $y(t) = c$ (Some constant

$\therefore dy/dt = 0$

$\therefore$ from (1) $0 + 5c = 15$

$c = \frac{15}{5} = 3$

ie $y_p = 3$ (equi).

CF : $y(t) = A \cdot e^{-\int 5 dt} = A e^{-5t + k}$

$-5t$

CF : $y(t) = A \cdot e^{-\int 5\,dt} = A\,e^{-5t+k}$
$= \underbrace{A \cdot e^{k}} \cdot e^{-5t}$

$y_c = y(t) = \lambda \cdot e^{-5t}$

$\therefore$ Differential equation of y is

$y(t) = y_p + y_c$

$y(t) = 3 + \lambda e^{-5t}$

From the initial condition

$y(0) = 8$

$3 + \lambda e^{-5(0)} = 8$

$3 + 1\lambda = 8$

$\lambda = 8 - 3 = 5$

$\therefore$ Required soln is $y(t) = 3 + 5e^{-5t}$ (ans).

(3) $\frac{dy}{dt} + 4y = 12$ given $y(0) = 5$.

PI : $0 + 4y_p = 12$

$\boxed{y_p = 3}$

CF: $y(t) = Ae^{-\int 4\,dt}$
$= A\,e^{-4t+k}$
$= A\,e^{k}\,e^{-4t}$

$y(c) = \lambda e^{-4t}$

$\therefore y(t) = y(p) + y(c)$
$= 3 + \lambda e^{-4t}$

$y(0) = 5$

$3 + \lambda e^{0} = 5$

$3 + \lambda = 5$

$\lambda = 2$

$\therefore y(t) = 3 + 2e^{-4t}$ (ans).

stability ⇒ convergent (with ↑se in time as $t \to \infty$ ⇒ $y(t) \Rightarrow$ equil)

unstable ⇒ divergent ($t \to \infty$ ⇒ $y(t)$ moves away from equilibrium)

$$\boxed{y(t) = 3 + 2e^{-4t}}$$

whenever as $t \to \infty$ $A \cdot e^{-at} \to 0$ then it is stable or convergent

as $t \to \infty$ $\quad 2e^{-4t} = \frac{2}{e^{4t}} \to 0$

$\therefore \quad y(t) = 3 + 0 = 3$ (equil)

ie as $t \to \infty$ $\quad y(t) \to 3$ (equilibrium)

$\therefore$ The time path is convergent or stable.