Lagrange Multiplier

Nmb values
$x = y$ units
$y = 1$

Q. Let the utility function be given by $U = xy$
and the budget constraint be given as
$100 - P_x \cdot x - P_y \cdot y = 0$

Find the demand function of x and y.

Lagrangian expression can be written as

$$L = (xy) + \lambda\left(100 - P_x \cdot x - P_y \cdot y\right)$$

For maximisation F.O.C requires

$$\frac{\partial L}{\partial x} = 0 \Rightarrow y + \lambda(-P_x) = 0$$
$$\Rightarrow \boxed{\lambda = y/P_x} \quad \text{— (1)}$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow x + \lambda(-P_y) = 0$$
$$\boxed{\lambda = x/P_y} \quad \text{— (2)}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow 100 - P_x \cdot x - P_y \cdot y = 0 \quad \text{— (3)}$$

Comparing (1) and (2)

$$\frac{y}{P_x} = \frac{x}{P_y}$$

$$\boxed{y = \frac{x}{P_y} \cdot P_x} \quad \text{— (4)}$$

Putting the value of 'y' in eq (3), we get:

$$100 - P_x \cdot x - \cancel{P_y}\left(\frac{x}{\cancel{P_y}} \cdot P_x\right) = 0$$

$$100 = P_x \cdot x + x P_x$$

$$100 = 2 P_x \cdot x$$

$100 = 2 P_x \cdot x$

$$\therefore\ \boxed{y = \frac{50}{\cancel{P_x} \cdot P_y} \cdot \cancel{P_x} = \frac{50}{P_y}}$$

$$\therefore\ \boxed{x = \frac{50}{P_x}}$$

$\therefore$ The demand function for x and y are resp,

D of homo of x & y = ?

$$x = \frac{50}{P_x} \quad \text{and} \quad y = \frac{50}{P_y}.$$

Q2. $U = x^{\alpha} y^{\beta}$ subject to budget constraint $P_x \cdot x + P_y \cdot y = M$.

Find the demand functions for x and y.

$$L = x^{\alpha} y^{\beta} + \lambda (P_x \cdot x - P_y \cdot y - M)$$

F.O.C's

$$\frac{\partial L}{\partial x} = 0 \Rightarrow \alpha x^{\alpha-1} \cdot y^{\beta} + \lambda (P_x) = 0$$

$$\Rightarrow \boxed{\lambda = -\frac{\alpha x^{\alpha-1} y^{\beta}}{P_x}} \quad \text{— (1)}$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow x^{\alpha} \cdot \beta y^{\beta-1} + \lambda P_y = 0$$

$$\Rightarrow \boxed{\lambda = -\frac{\beta x^{\alpha} y^{\beta-1}}{P_y}} \quad \text{— (2)}$$

$$\frac{\partial L}{\partial \lambda} = 0$$

$\frac{\partial L}{\partial \lambda} = 0$

$\longrightarrow \Rightarrow P_x \cdot x + P_y \cdot y - M = 0$

or, $\boxed{P_x \cdot x + P_y \cdot y = M}$ — (3)

Comparing (1) and (2), we get,

$$\frac{\cancel{\lambda}\,\alpha x^{\alpha-1} y^{\beta}}{P_x} = \frac{\cancel{\lambda}\,\beta x^{\alpha} y^{\beta-1}}{P_y}$$

$$\Rightarrow \frac{\alpha}{\beta} \frac{x^{\alpha-1}}{x^{\alpha}} = \frac{y^{\beta-1}}{y^{\beta}} \cdot \frac{P_x}{P_y}$$

$$\Rightarrow \frac{\alpha}{\beta} \frac{1}{x} = \frac{P_x}{P_y} \cdot \frac{1}{y}$$

$$\Rightarrow \boxed{x = \frac{\alpha}{\beta} \cdot \frac{P_y}{P_x} \cdot y} \text{ — (4)}$$

Substituting value of 'x' in eq (3) (budget contraint)

$$P_x \cdot x + P_y \cdot y = M$$

or, $\cancel{P_x} \cdot \left(\frac{\alpha}{\beta} \cdot \frac{P_y}{\cancel{P_x}} \cdot y\right) + P_y \cdot y = M$

or, $\frac{\alpha}{\beta} \cdot P_y \cdot y + P_y \cdot y = M$

or, $\frac{\alpha}{\beta} \cdot P_y \cdot y + P_y \cdot y$

or, $P_y \cdot y \left(\frac{\alpha}{\beta} + 1 \right) = M$

or, $P_y \cdot y \left(\frac{\alpha + \beta}{\beta} \right) = M$

or, $\boxed{y = \frac{M \cdot \beta}{(\alpha+\beta) \cdot P_y}}$

$\therefore x = \frac{\alpha}{\beta} \frac{P_y}{P_x} \cdot y = \frac{\alpha}{\cancel{\beta}} \cdot \frac{\cancel{P_y}}{P_x} \left(\frac{M \cancel{\beta}}{(\alpha+\beta) \cancel{P_y}} \right)$

$$\boxed{x = \frac{\alpha M}{(\alpha+\beta) P_x}}$$

x and y are demand function also called the Marshallian or Ordinary demand function.

$$\left.\begin{array}{l} x = f(P_x, P_y, M) \\ y = f(P_x, P_y, M) \end{array}\right\} \text{Marshallian or Ordinary demand function}$$

Minimisation Problem

Given utility function, $U = q_1 q_2$
and budget fn: $M = p_1 \cdot q_1 + p_2 q_2$.
Ex

$\boxed{\text{Objective is to minimise } p_1 q_1 + p_2 q_2 \quad \text{s.bj to } \bar{U} = q_1 q_2 \text{ (constraint)}.}$

$\bar{U} - q_1 q_2 = 0$

Lagrangian expression is

$$L = (p_1 q_1 + p_2 q_2) + \lambda (\bar{U} - q_1 q_2)$$

F.o.c ~~~

$$\frac{\partial L}{\partial q_1} = 0 \Rightarrow p_1 + \lambda(-q_2) = 0$$
$$\Rightarrow \lambda = p_1/q_2 \quad \text{— (1)}$$

$$\frac{\partial L}{\partial q_2} = 0 \Rightarrow p_2 - \lambda q_1 = 0$$
$$\boxed{\lambda = p_2/q_1} \quad \text{— (2)}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow \boxed{\bar{U} = q_1 q_2} \quad \text{— (3)}$$

Comparing (1) and (2)

$p_1 \quad p_2$

Comparing ① ... ②

$$\frac{P_1}{q_2} = \frac{P_2}{q_1}$$

$$\boxed{q_1 = \frac{P_2}{P_1} \cdot q_2} \quad \text{— } ④$$

Let us substitute ④ in ③ we get

$$\bar{U} = q_1 q_2$$

$$\bar{U} = \frac{P_2}{P_1} \cdot q_2 \cdot q_2$$

$$\bar{U} = \frac{P_2}{P_1} q_2^2$$

$$q_2^2 = \frac{P_1 \bar{U}}{P_2}$$

or, $\boxed{q_2 = \sqrt{\frac{P_1 \bar{U}}{P_2}}}$

$$\therefore q_1 = \frac{P_2}{P_1} q_2 = \frac{P_2}{P_1} \sqrt{\frac{P_1 \bar{U}}{P_2}}$$

$$\boxed{q_1 = \sqrt{\frac{P_2 \bar{U}}{P_1}}}$$

q_1 and q_2 $(P_2, P_3, \bar{U}) \rightarrow$ compensated demand.

q_1 and q_2
$\rightarrow f(P_2, P_3, \bar{U}) \rightarrow$ compensated demand function.

Summary:

1. $L = f(\quad) + \lambda[\quad]$
2. f.o.c.s : $\partial L/\partial x = 0$ $\quad \partial L/\partial y = 0$ $\quad \cancel{\partial L/\partial \lambda} = 0$
   $\downarrow$ $\lambda = ?$ $\quad \downarrow$ $\lambda = ?$
3. compare 'λ' $\Rightarrow$ establish x as fn of y
   $\boxed{x = (\quad) y}$
4. Substitute this 'x' in (3) f.o.c. and get 'y' as demand function
   $y = f(P_x, P_y, M)$
   or $y = f(P_x, P_y, \bar{U})$
5. Now put this 'y' in 3rd point and get demand function for X

— * —

TRY

Q Construct ordinary and Compensated demand functions for the utility function $\boxed{U = 2q_1q_2 + q_2}$

function $\boxed{U = \ldots q_1 \ldots q_2 \ldots}$

Budget fn is $M = P_1 q_1 + P_2 q_2$

Degree of Homogeneity

Q $Z = xy$ Find the degree of homogeneity.

Let us change 'x' and 'y' by λ proportion then we get $(\lambda x)(\lambda y)$

$= \lambda^2 xy$

$= \lambda^{2} Z$ degree of homogeneity.

That means, degree of homogeneity is 2.

IRS

Q $\Rightarrow$ $Q = x^{0.4} . y^{0.6}$ Find degree of homogeinity.

Let us change 'x' and 'y' by λ proportion, then we get,

$(\lambda x)^{0.4} (\lambda y)^{0.6}$

$= \lambda^{0.4} x^{0.4} \lambda^{0.6} y^{0.6}$

$= \lambda^{0.4} x^{0.4} \lambda^{0.6} y^{0.6}$

$= \lambda^{0.4+0.6} x^{0.4} y^{0.6}$

$= \lambda \, x^{0.4} y^{0.6}$

CRS $\longrightarrow$ $= \lambda^{1} \cdot Q$

$\therefore$ Degree of Homogeneity is 1.

Increasing returns to scale (IRS)

$$f(\lambda x, \lambda y) < \lambda Q$$

Constant returns to scale (CRS)

$$f(\lambda x, \lambda y) = \lambda Q$$

Decreasing returns to scale (DRS)

$$f(\lambda x, \lambda y) > \lambda Q$$

Ex: $Q = A \cdot x^{\alpha} y^{\beta}$ where α, β are const A.

Let us change x and y by 'λ' proportion, we get,

$$A \cdot (\lambda x)^{\alpha} (\lambda y)^{\beta}$$

$$= A \, \lambda^{\alpha} x^{\alpha} \cdot \lambda^{\beta} y^{\beta}$$

$= A \lambda^{\alpha} x^{\alpha} \cdot \lambda^{\beta} y^{\beta}$

$= \underbrace{A x^{\alpha} y^{\beta}}_{Q} \lambda^{\alpha+\beta}$

$= \lambda^{(\alpha+\beta)} \cdot Q$

$\therefore$ Degree of homogeneity is $(\alpha+\beta)$.

1. If $(\alpha+\beta) > 1 \Rightarrow$ IRS
2. If $(\alpha+\beta) = 1 \Rightarrow$ CRS
3. If $(\alpha+\beta) < 1 \Rightarrow$ DRS.