08 February 2024 07:38 PM

Interval Estimation.

Digression:

(1) We have Normal distribution with parameter μ, σ^2

ie $X \sim N(\mu, \sigma^2)$.

(2) $Z = \dfrac{X - E(x)}{SE(x)} = \dfrac{X - \mu}{\sigma} = Z$ or τ

where Z or τ is the std normal variable.

$Z \sim N(0, 1)$

(3) sum of the square of std normal variable $\to \chi^2$.

ie $\sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} \left(\dfrac{x_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$

(This is chi-square distribution with degree of freedom 'n')

(4) Student's t-distribution:

Let $x \sim N(0,1)$

and $y \sim \chi^2_{(n)}$ } also x and y are independent random variables and $y \sim \chi^2_{(n)}$ } are independent random variables.

then $t = \dfrac{x}{\sqrt{y/n}} \sim t_{(n)}$

⑤ F-distribution.

Let $x \sim \chi^2_{(m)}$ and $y \sim \chi^2_{(n)}$

and x and y are independent random variable

then $F = \dfrac{x/m}{y/n} \sim F_{(m,n)}$

RSWR, RSWOR.

$E(\bar{x}) = \mu$

$SE(\bar{x}) = \dfrac{\sigma}{\sqrt{n}}$

Case 1: Confidence limits to μ when $\boxed{\sigma \text{ is KNOWN}}$

Let the independent random sample obs be $x_1, x_2 \ldots x_n$, then $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the sample mean

then the test statistic in this case is

$$\tau = \frac{\bar{x} - E(\bar{x})}{SE(\bar{x})} = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

at α level of significance

then $1-\alpha \Rightarrow$ acceptance region.

← confidence interval limit

$$P\left[\bar{x} - \sigma_\alpha \ldots \le \mu \le \bar{x} + \sigma_\alpha \ldots\right] = 1 - \alpha$$

$$P\left[\bar{x} - \frac{\sigma}{\sqrt{n}} \tau_{\alpha/2} \le \mu \le \bar{x} + \frac{\sigma}{\sqrt{n}} \tau_{\alpha/2}\right] = 1 - \alpha$$

$\tau_\alpha = \tau_{0.05}$ or $\tau_{0.01}$ or $\tau_{0.1}$ $\qquad \tau_{0.025}$, $\tau_{0.005}$

$\alpha = 5\%$ or 1% or 10% | $\alpha/2$ $\quad \frac{0.05}{2} = 0.025$, $\frac{0.01}{2} = 0.005$: . . . .

$0.05 \quad 0.01 \quad 0.1$

$1-\alpha$	95%	99%	90%
	0.95	0.99	0.9

Case 2: Confidence limit to μ, when $\boxed{\sigma \text{ is UNKNOWN.}}$

Let the independent observation be $x_1, x_2, \dots, x_n$ and let $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$

biased ~~sd~~ variance is $(s)^2 = \frac{1}{n} \sum (x_i - \bar{x})^2 \rightarrow n s^2 = \sum (x_i - \bar{x})^2$

Unbiased ~~sd~~ variance is $(s')^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2 \rightarrow (n-1) s'^2 = \sum (x_i - \bar{x})^2$

What is the relation between s and s'?

From the formula, $n s^2 = (n-1) s'^2$

or, $\frac{s^2}{n-1} = \frac{s'^2}{n}$

or, $\frac{s}{\sqrt{n-1}} = \frac{s'}{\sqrt{n}}$

$\therefore$ The test statistic is

$$t = \frac{\bar{x} - \mu}{s'/\sqrt{n}} \sim t_{(n-1)}.$$

$\Rightarrow$ this follows t-distribution with df $(n-1)$.

$$P\left[\underbrace{\bar{x} - \frac{s'}{\sqrt{n}} t_{\frac{\alpha}{2}, n-1}} \le \mu \le \underbrace{\bar{x} + \frac{s'}{\sqrt{n}} t_{\alpha/2, n-1}}\right] = 1 - \alpha$$

Case 3: Confidence limit to $\boxed{\sigma}$ when $\boxed{\mu \text{ is known}}$

$$\frac{\sum_{i=1}^{n} (x_i - \mu)^2}{\sigma^2} \sim \chi^2_{(n)}$$

$$P\left[\underbrace{\frac{\sum (x_i - \mu)^2}{\chi^2_{\alpha/2, n}}} \le \sigma^2 \le \underbrace{\frac{\sum (x_i - \mu)^2}{\chi^2_{1-\frac{\alpha}{2}, n}}}\right] = 1 - \alpha$$

case 4: Confidence limit to $\boxed{\sigma}$ when $\boxed{\mu \text{ is unknown}}$

$$\sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\sigma^2} = \frac{(n-1)s'^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

$$P\left[\underbrace{\frac{(n-1)s'^2}{\chi^2_{\alpha/2, n-1}}} \le \sigma^2 \le \underbrace{\frac{(n-1)s'^2}{\chi^2_{1-\frac{\alpha}{2}, n-1}}}\right] = 1 - \alpha$$

Ex: Suppose that a random sample of size 10, drawn from a normal population has mean 40 and s.d 12. Find 99% Confidence limits for the population mean (μ). (Given $t_{0.005} = 3.25$ for 9 d.f)

Soln: $n = 10$ $\bar{x} = 40$ and $s = 12$

$1 - \alpha = \frac{99}{100} = 0.99$

$\therefore \alpha = 0.01$.

$\alpha/2 = 0.005$

... Confidence

Soln: $n = 10$ $\bar{x} = 40$ ... $\frac{\ }{100}$

$\therefore \alpha = 0.01$.
$\alpha/2 = 0.005$

We have to find the confidence limit to μ, when σ is unknown

$$\therefore P\left[\bar{x} - t_{\alpha/2, n-1}\frac{s}{\sqrt{n-1}} \le \mu \le \bar{x} + t_{\frac{\alpha}{2}, n-1}\right] = 0.99$$

$$P\left[40 - t_{0.005, 9}\cdot\frac{12}{\sqrt{9}} \le \mu \le 40 + t_{0.005, 9}\cdot\frac{12}{\sqrt{9}}\right] = 0.99$$

$$P\left[40 - 3.25 \times 4 \le \mu \le 40 + 3.25 \times 4\right] = 0.99$$

$$P\left[27 \le \mu \le 53\right] = 0.99$$

Ex2: Suppose the s.d of a random sample of size 20 from a normal population is 12.

Find 95% confidence interval for the population variance. [Given that for 19 d.f $\chi^2_{0.975} = 8.907$ and $\chi^2_{0.025} = 32.852$]

Solution: Given $n = 20$, $s = 12$, $1 - \alpha = \frac{95}{100} = 0.95$

$s^2 = 144$ $\therefore \alpha = 0.05$ (= 5%).

$\frac{\alpha}{2} = 0.025$

$1 - \alpha/2 = 1 - 0.025$
$= 0.975$

$$P\left[\frac{1}{\chi^2_{\frac{\alpha}{2}, n-1}}\cdot ns^2 \le \sigma^2 \le \frac{ns^2}{\chi^2_{1-\frac{\alpha}{2}, n-1}}\right] = 1 - \alpha$$

$$P\left[\frac{20 \times 144}{\chi^2_{0.025, 19}} \le \sigma^2 \le \frac{20 \times 144}{\chi^2_{0.975, 19}}\right] = 0.95$$

$$P\left[\frac{20 \times 144}{\ } \le \sigma^2 \le \frac{20 \times 144}{\ }\right] = 0.95$$

$$P\left[\frac{20\times 144}{32.852} \le \sigma^2 \le \frac{20\times 144}{8.907}\right] = 0.95$$

$$P\left[\,87.66 \le \sigma^2 \le 323.341\,\right] = 0.95$$