DIFF CAL 3 25.4.2024

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Diff Cal Green Qu...

Diff calculus

Sem III ..

↓ Diff ↙ One step backward

① Let f be a continuously differentiable function, such that $\int_0^{2x^2} f(t)dt = e^{\cos x^2}$ for all $x \in (0, \infty)$. The value of $f'(\pi)$ is ________.

Leibnitz's Rule

$$f(2x^2)\frac{d}{dx}(2x^2) = -(\sin x^2)(2x)\, e^{\cos x^2}$$

$$f(2x^2)\, 4x = \quad 2\, f(2x^2) = -\sin x^2 \cdot e^{\cos x^2}$$

$$2 f(2x^2)\, 4x = -\cos x^2 (2x) e^{\cos x^2} + (-\sin x^2)(-\sin x^2)\, 2x \cdot e^{\cos x^2}$$

$f'(\pi) \rightarrow$ put $x = \sqrt{\frac{\pi}{2}}$

$$8\sqrt{\frac{\pi}{2}}\, f'(\pi) = 0 + (-1)(-1)\, 2\cdot\sqrt{\frac{\pi}{2}}\, e^0$$

$$\Rightarrow 8\cancel{\sqrt{\frac{\pi}{2}}}\, f'(\pi) = 2\cancel{\sqrt{\frac{\pi}{2}}}$$

$$f'(\pi) = 0.25$$

more vale => 1

9. *The set of points at which the function* $f(x, y) = x^4 + y^4 - x^2 - y^2 + 1$, $(x,y) \in \mathbb{R}^2$ *attains local maximum is__________.* **(NAT)**

$f = x^4 + y^4 - x^2 - y^2 + 1$

$f_x = 4x^3 - 2x \qquad f_y = 4y^3 - 2y$

$f_{xx} = 12x^2 - 2 \qquad f_{yy} = 12y^2 - 2$

$f_{xy} = 0$

$f_x = f_y = 0 \qquad x = 0, \pm\frac{1}{\sqrt{2}} \qquad y = 0, \pm\frac{1}{\sqrt{2}}$

Stationary pt $(0,0)$ $\left(0, \pm\frac{1}{\sqrt{2}}\right)$, $\left(\pm\frac{1}{\sqrt{2}}, 0\right)$ $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

pt	f_{xx}	f_{yy}	$f_{xx}f_{yy} - f_{xy}^2$	Behav
$0,0$	<0	<0	>0	max
$0, \pm\frac{1}{\sqrt{2}}$	<0	>0	<0	Neither
$\pm\frac{1}{\sqrt{2}}, 0$	>0	<0	<0	neither
$\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}$	>0	>0	>0	min

$\frac{2(x^3+y^3)}{x^2+y^2} \qquad x, y \neq 0, 0$

$x, y = 0, 0$

$$\frac{2(x+y)}{x^2+2y} \qquad x, y = 0, 0$$

0

3) 7. Let $f(x,y)=\begin{cases}\dfrac{2(x^3+y^3)}{x^2+2y}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$. Show that, the first order partial derivatives of f with respect to x and y exist at (0, 0). Also, show that f is not continuous at (0, 0).

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{2h^3}{h^2} - 0\right] = 2$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{1}{k}\left[\frac{2k^3}{2k} - 0\right]$$

$$= 0$$

for order PD of x exist at $(0,0)$

Now, checking continuity at f at $(0,0)$ → take $(x,y)\to(0,0)$

$$y = -\frac{x^2}{2} + mx^3$$

$$\therefore f\left(x, -\frac{x^2}{2} + mx^3\right) = \frac{2x^3\left(1 + \left(-\frac{x}{2} + mx^2\right)\right)}{2mx^3}$$

then, $\lim_{x\to 0} f\left(x, -\frac{x^2}{2} + mx^3\right) = \frac{1}{m}$

Hence, $\lim_{x,y\to 0,0} f(x,y)$ does not exist

hence function is not cont

Subjective.

4) 8. Let $f: [a, b] \to \mathbb{R}$ be a continuous function, such that f is differentiable in (a,c) and (c, b), $a < c < b$. If $\lim_{x\to c} f'(x)$ exists, then prove that f is differentiable at 'c' and $f'(c) = \lim_{x\to c} f'(x)$.

$\lim_{x\to c} f'(x)$ exist. → $\lim_{x\to c^-} f'(x)$ & $\lim_{x\to c^+} f'(x)$ $=$

[illegible] ... [illegible] in $[a,c)$ $\boxed{a < c}$

As f is differentiable in (a,c) $\boxed{a<c}$

$$\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c} = f'(c^-) \text{ exists}$$

Also f is diff (c,b) $\rightarrow$ $\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c} = f'(c^+)$

As, $\lim_{x\to c} f'(x)$ exists so, $\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}$

$f'(c^-) = f'(c^+)$ $= \lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}$

So, $f(x)$ is dif at $x=c$

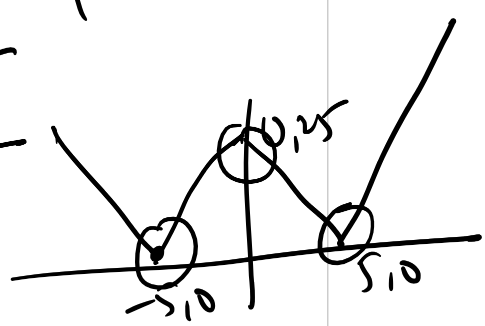
$$\boxed{f'(c) = \lim_{x\to c} f'(x)}$$ Ans

(5) 13. *Let $f(x)=|x^2-25|$ for all $x\in\mathbb{R}$. The total number of points of $\mathbb{R}$ at which f attains a local extremum (minimum or maximum) is* **(MCQ)**

(a) 1 *(b) 2* *(c) 3* *(d) 4*

$f'(x)=0 \Rightarrow \frac{(x^2-25)\,2x}{|x^2-25|}=0$

3 extremum local value $x=0, 5, -5$



$$\frac{\partial f}{\partial x} = \sum k (x^2-y^2)^{k-1} (2x)$$

$$\partial f/\partial y = \sum k (x^2-y^2)^{k-1} (-2y)$$

6 Let $f(x, y) = \sum_{k=1}^{10} (x^2 - y^2)^k$ for all $(x, y) \in \mathbb{R}^2$. Then for all $(x, y) \in \mathbb{R}^2$, **(MCQ)**

(a) $x\frac{\partial f}{\partial x}(x, y) - y\frac{\partial f}{\partial y}(x, y) = 0$

(b) $x\frac{\partial f}{\partial x}(x, y) + y\frac{\partial f}{\partial y}(x, y) = 0$

(c) $y\frac{\partial f}{\partial x}(x, y) - x\frac{\partial f}{\partial y}(x, y) = 0$

(d) $y\frac{\partial f}{\partial x}(x, y) + x\frac{\partial f}{\partial y}(x, y) = 0$ ✓

So, $\sum k (x^2-y^2)^{k-1} = \frac{1}{2x}\frac{\partial f}{\partial x}$

$$\sum k (x^2-y^2)^{k-1} = -\frac{1}{2y}\frac{\partial f}{\partial y}$$

$$\frac{1}{2x}\partial f/\partial x = -\frac{1}{2y}\frac{\partial f}{\partial y}$$

$$\Rightarrow \quad y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0$$

Also if $x = 0$ or $y = 0$ $\quad y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0$

hold trivially

hence ... 0

7 The value of $\alpha \in \mathbb{R}$ for which the curves $x^2+\alpha y^2=1$ and $y=x^2$ intersect orthogonally is (MCQ)

(a) -2 (b) $\frac{-1}{2}$ (c) $\frac{1}{2}$ (d) 2

$$\boxed{\alpha_1 \alpha_2 = -1}$$

α_1 , α_2

$x^2+\alpha y^2=1 \Rightarrow 2x + 2\alpha y \frac{dy}{dx} = 0 \quad \frac{dy}{dx} = \alpha_1 = -\frac{x}{\alpha y}$

$\Rightarrow y = x^2 \Rightarrow \frac{dy}{dx} = 2x \quad \boxed{\alpha_2 = 2x}$

$\alpha_1 \alpha_2 = -1 \quad \alpha_1 = -\frac{1}{2x}$

$\left(-\frac{x}{\alpha y}\right)(2x) = -1$

$\Rightarrow y = x^2, \quad \frac{2x^2}{\alpha x^2} = 1$

$$\boxed{\alpha = 2}$$

$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim \frac{h}{|h|} \sqrt{h^2 - 0}$

$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{h}{|h|}\sqrt{h^2} - 0}{h}$

$= \lim_{h \to 0} \frac{|h|}{|h|} = 1$

$1 + 0 \Rightarrow 1$

8 *For all* $(x, y) \in \mathbb{R}^2$, *let* $f(x, y) = \begin{cases} \frac{x}{|x|}\sqrt{x^2 + y^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$. *Then,* $\frac{\partial f}{\partial x}(0,0) + \frac{\partial f}{\partial y}(0,0)$ *equals* **(MCQ)**

(a) -1 *(b) 0* *(c) 1* *(d) 2*

$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{0 - 0}{k} = 0$

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $f\left(\frac{1}{2^n}\right) = 0$, *for all* $n \in \mathbb{N}$. *Show that* $f'(0) = 0 = f''(0)$.

$n \to \infty$, $\frac{1}{2^n} \to 0$

$f(0) = 0$

Taylor's series neighbourhood of 0 upto 3rd order

$$f\left(\frac{1}{2^n}\right) = f(0) + \frac{1}{2^n} f'(0) + \left(\frac{1}{2^n}\right)^2 \frac{1}{2!} f''(0)$$

or, $0 = 0 + \frac{1}{2^n} f'(0) + \frac{1}{2^{2n+1}} f''(0)$

$$\boxed{f'(0) = f''(0) = 0}$$

21. Suppose that the dependent variables z and w are functions of the independent variables **x and y**, defined by the equations $f(x, y, z, w) = 0$ and $g(x, y, z, w) = 0$, where $f_z g_w - f_w g_z = 1$. Which one of the following is correct? **(MCQ)**

(a) $z_x = f_w g_x - f_x g_w$ (b) $z_x = f_x g_w - f_w g_x$ (c) $z_x = f_z g_x - f_x g_z$ (d) $z_x = f_z g_w - f_z g_x$

22. *For what real values of x and y, does the integral* $\int_x^y (6-t-t^2)dt$ *attain its maximum?* **(MCQ)**

(a) $x = -3, y = 2$ (b) $x = 2, y = 3$ (c) $x = -2, y = 2$ (d) $x = -3, y = 4$

28. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be defined by* $f(x, y) = \begin{cases} \left(1 + \dfrac{x}{y}\right)^2, & y \neq 0 \\ 0, & y = 0 \end{cases}$

If the directional derivative of f at (0, 0) exists along the direction $\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$, *where* $\sin\alpha \neq 0$, *then the value of* $\cot\,\alpha$ *is* ________ ***(NAT)***

31. *The function $f(x, y)=3x^2y + 4y^3 - 3x^2 - 12y^2 + 1$ has a saddle point at* **(MCQ)**

(a) (0, 0) *(b) (0, 2)* *(c) (1, 1)* ***(d)** (-2, 1)*

33. *Let f be a real valued function defined by* $f(x, y) = 2\, \ell n\left(x^2 y^2 e^{\frac{y}{x}}\right)$, $x > 0, y > 0.$

Then, the value of $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y}$ *at any point (x,y), where x>0, y>0, is* ______________ **(NAT)**

34. *The maximum value of $f(x, y) = x^2 + 2y^2$ subject to constraint $y - x^2 + 1 = 0$ is* ______ **(NAT)**

36. $f(x)=\begin{cases}1+x & , \text{ if } x<0 \\ (1-x)(px+q) & , \text{ if } x\geq 0\end{cases}$ *satisfies the assumptions of Rolle's theorem in the* ***interval*** *[-1, 1], then the ordered pair (p, q) is* ***(MCQ)***

(a) (2, -1) *(b) (-2, -1)* *(c) (-2, 1)* *(d) (2, 1)*