Linear Second Order Differential Eqns with constant coeffs:

Form: $\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = Q$ [where P_1, P_2 are constants, Q is a fn of x]

(i) Consider the homogeneous part: [For CF]

$$\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0.$$

Assume a trial soln: $y = e^{mx}$.

A.E: $m^2 + P_1 m + P_2 = 0 \Rightarrow$ Two roots m_1, m_2

(*) $m_1 \neq m_2$ & real.

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

(*) $\boxed{m_1 = m_2}$ & real $\Rightarrow y_c = c_1 e^{mx} + c_2 e^{mx} = (c_1 + c_2) e^{mx} = k \cdot e^{mx}$

$$y_c = c_1 \boxed{e^{mx}} + c_2 \boxed{x e^{mx}} = (c_1 + c_2 x) e^{mx}$$

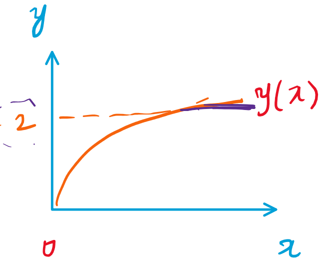
(*) m_1, m_2 are complex $= \alpha \pm i\beta$.

$$y_c = e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x]$$

(ii) Find the particular Integral (PI)

[Why ?] Final soln: $y = y_c + y_p$.

Eg: $\frac{d^2y}{dx^2} + 4 \cdot \frac{dy}{dx} - 3y = 6$ = RHS is also constant. Soln: $y = y(x)$



If $\frac{d^2y}{dx^2} = \frac{dy}{dx} = 0$, $\Rightarrow y = -2 \Rightarrow$ "Stable value" [long run value of y]

Homogeneous: $\frac{d^2y}{dx^2} + 4 \cdot \frac{dy}{dx} - 3y = 0$

$$\frac{dx^2}{} \quad \frac{}{dx} - y = 0$$

"Stable value", $\frac{d^2y}{dx^2} = \frac{dy}{dx} = 0 \Rightarrow \boxed{y = 0}$

Case I: If Q is a polynomial in 'x'.

Q. Solve: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = x^2$ ... (i)

$\frac{d^2y}{dx^2} = \frac{dy}{dx} = 0$

$-4y = x^2$

$\boxed{y = -\frac{x^2}{4}}$

Find CF: Let e^{mx} be a trial soln

A.E: $m^2 + 3m - 4 = 0$.

$m = -4, 1$ .... Real & unequal.

$\therefore y_c = c_1 \cdot e^{-4x} + c_2 e^x$.

Finding PI: $y_p = \boxed{A_0 + A_1 x + A_2 x^2} \rightarrow$ [Find A_0, A_1, A_2]

Eg: RHS: $x^3 + 3x^2$, $y_p = A_0 + A_1 x + A_2 x^2 + A_3 x^3$.

→ " Long Run Trajectory"

y_p satisfies (i):-

$y_p' = 2x A_2 + A_1 \qquad y_p'' = 2A_2$

$\Rightarrow (2A_2) + 3(2x A_2 + A_1) - 4(A_0 + A_1 x + A_2 x^2) = x^2$

$(2A_2 + 3A_1 - 4A_0) + (6A_2 - 4A_1)x - 4A_2 x^2 = x^2$

Compare coeff: x^2: $-4A_2 = 1 \Rightarrow A_2 = -1/4$.

x: $6A_2 - 4A_1 = 0. \Rightarrow A_1 = -3/8$.

const: $2A_2 + 3A_1 - 4A_0 = 0 \Rightarrow A_0 = -13/32$

$y_p = -\frac{13}{32} - \frac{3}{8}x - \frac{1}{4}x^2$

$\therefore y = c_1 e^{-4x} + c_2 e^{x}$ ...

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$$\therefore \boxed{y = c_1 e^{-4x} + c_2 e^{x} - \frac{13}{32} - \frac{3}{8}x - \frac{1}{4}x^2}$$

Case II: If Q is exponential: $Q(x) = e^{\alpha x}$.

Q. Solve: $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 4y = 5e^{x}$ ——— (i).

C.F: $y_c = c_1 e^{-4x} + c_2 e^{x}$

For P.I: Let $y_p = e^{\alpha x + \beta} = e^{\beta} \cdot e^{\alpha x} = A_0 e^{x}$.

Let $y_p = A_0 x e^{x}$

$y_p' = A_0 e^{x} + A_1 x e^{x}$

$y_p'' = A_0 e^{x} + A_1 [e^{x} + x e^{x}] = 2A_0 e^{x} + A_0 x e^{x}$

Replace in (i): -

$(2A_0 e^{x} + \cancel{A_0 x} e^{x}) + 3(A_0 e^{x} + \cancel{A_1 x} e^{x}) - 4\cancel{A_0 x} e^{x} = 5e^{x}$

$5A_0 e^{x} = 5e^{x} \Rightarrow A_0 = 1$

$\therefore y_p = x e^{x}$

Complete soln: $\boxed{y = c_1 e^{-4x} + c_2 e^{x} + x e^{x}}$