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Gauss Test

If $\sum u_n$ is a series of +ve terms s.t

$$\boxed{\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma_n}{n^p}},\quad \alpha > 0,\ p > 1 \ \& \ \{\gamma_n\} \text{ is a bounded seq.}$$

(i) $\alpha \neq 1 \Rightarrow \sum u_n$ converges if $\alpha > 1$ & diverges if $\alpha < 1$

(ii) $\alpha = 1 \Rightarrow \sum u_n$ converges if $\beta > 1$ & diverges if $\beta \leq 1$

Q. $$\sum u_n = \overset{\nearrow u_1}{\left(\frac{1}{2}\right)^2} + \overset{\nearrow u_2}{\left(\frac{1\cdot 3}{2\cdot 4}\right)^2} + \overset{\nearrow u_3}{\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2} + \cdots\cdots$$

$$u_n = \left\{\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right\}^2.$$

$$u_{n+1} = \left\{\frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{2\cdot 4\cdot 6\cdots(2n)(2n+2)}\right\}^2$$

$\sum \frac{1}{n^p}$.

- Comparision Test $(\sum v_n)$
- Cauchy Root Test $(u_n)^{1/n}$ ✗
- ✗ D-Alembert's Ratio Test
- ✗ Raabe's Test
- ✗ Logarithmic Test
- Gauss Test.

$\left.\right\} \frac{u_n}{u_{n+1}}$

$$\boxed{\frac{u_n}{u_{n+1}}} = \left(\frac{2n+2}{2n+1}\right)^2.$$

Ratio Test:

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l \begin{cases} \to < 1 \Rightarrow \text{convergent} \\ \to < 1 \Rightarrow \text{Divergent} \\ \to = 1 \Rightarrow \text{Inconclusive} \end{cases}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l \begin{cases} \to > 1 \Rightarrow \text{convergent} \\ \to < 1 \Rightarrow \text{Divergent} \\ \to = 1 \Rightarrow \text{Inconclusive} \end{cases}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$$ [Ratio Test not applicable].

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \lim_{n\to\infty} n \cdot \frac{4n+3}{(2n+1)^2}$$

$$= 1$$ [Raabe's Test fails].

$$\lim_{n\to\infty} n \log\left(\frac{u_n}{u_{n+1}}\right) \cdots$$ [Logarithmic Test is not applicable]

Applying Gauss Test:

Expand using Binomial Th

u_n = $(2n+2)^2$ $(1+\frac{1}{\ }\,)^2$. 2 -2)

Test :

Binomial Th

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2} = \left(1+\frac{1}{n}\right)^2 \left(1+\frac{1}{2n}\right)^{-2}$$

Now, $\left(1+\frac{1}{2n}\right)^{-2} = 1 - 2\left(\frac{1}{2n}\right) + 3\left(\frac{1}{2n}\right)^2 + \cdots = 1 - \frac{1}{n} + \frac{3}{4n^2} \cdots$

$n \geq 1 \Rightarrow \frac{1}{2n} < 1.$

Recap: [Binomial Expansions].

(i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots \infty$

(ii) $(1-x)^{-1} = 1 + x + x^2 \cdots \infty$

(iii) $(1+x)^{-2} = 1 - 2x + 3x^2 \cdots \infty$

(iv) $(1-x)^{-2} = 1 + 2x + 3x^2 \cdots \infty$

Valid only under $|x| < 1$ $\Rightarrow -1 < x < 1$

$$\therefore \frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)^2 \left(1+\frac{1}{2n}\right)^{-2}$$

$$= \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\left(1 - \frac{1}{n} + \frac{3}{4n^2} \cdots\cdots\right)$$

$$= 1 + \left\{\left(\frac{2}{n}\right) - \left(\frac{1}{n}\right)\right\} + \left(\frac{3}{4n^2}\right) + \left(\frac{1}{n^2}\right) - \frac{2}{n^2} \cdots\cdots$$

$$= \underbrace{1 + \frac{1}{n} - \frac{1}{4n^2}}_{\alpha + \frac{\beta}{n} + \frac{\gamma_n}{n^p}} + \cdots \left(\text{higher powers of } \frac{1}{n}\right).$$

$\alpha > 0$, $p > 1$, $\{\gamma_n\}$ should be bounded seq.

↳ $\alpha = 1$, $\beta = 1$ $\Rightarrow$ $\sum u_n$ is divergent.

Summarizing:

$\sum u_n$ is a series of positive terms.

↳ Comparision Test: Construct $\sum v_n$ where v_n is from p-series.

p-series.

↳ Cauchy Root Test: $(u_n)^{\frac{1}{n}}$ ... useful for powers of n expressions.

↳ Ratio Test

Raabe's Test

↙ ↘

Logarithmic Test   Gauss Test.

$\Bigg\}\ \frac{u_n}{u_{n+1}}$.

Q. $\Sigma u_n = x\log x + x^2\log 2x + x^3\log 3x + \ldots\ldots$

Σu_n is convergent if $\overset{\times}{(a)}\ x > 1$ $\overset{\rho}{(b)}\ x \geq 1$

(c) $x < 1$ (d) $x \leq 1$

$u_n = x^n \log nx$. $\Rightarrow (u_n)^{\frac{1}{n}} = x\cdot[\log nx]^{\frac{1}{n}}$.

$u_{n+1} = x^{n+1}\log(n+1)x$. $\lim_{n\to\infty}$ (∞^0) ×

$\therefore \frac{u_n}{u_{n+1}} = \left[\frac{\log nx}{\log(n+1)x}\right]\cdot\frac{1}{x}$.

$\frac{u_n}{u_{n+1}} - 1 = \frac{\log nx - x\log(n+1)x}{x\log(n+1)x}$

$\frac{u_{n+1}}{u_n} = \left[\frac{\log(n+1)x}{\log nx}\right]\cdot x$.

$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = x\lim_{n\to\infty}\frac{\log(n+1)x}{\log nx}$. $\left[\frac{\infty}{\infty}\right]$.

L·Hopital's Rule :-

$$= x\cdot\lim_{n\to\infty}\frac{\frac{1}{(n+1)\cancel{x}}\cdot\cancel{x}}{\frac{1}{n\cancel{x}}\cdot\cancel{x}}$$

$$= x \cdot \lim_{n\to\infty} \frac{n}{n+1} = x.$$

(i) $x < 1 \Rightarrow$ Convergent

$x > 1 \Rightarrow$ Divergent

$x = 1 \Rightarrow$ Test inconclusive

Check $x = 1$; use Raabe's Test:

$$\frac{u_n}{u_{n+1}} = \left[\frac{\log nx}{\log (n+1)x}\right] \cdot \frac{1}{x}$$

Put $x = 1$. $\quad \frac{u_n}{u_{n+1}} = \frac{\log n}{\log(n+1)}$

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \lim_{n\to\infty} n\left[\frac{\log n - \log(n+1)}{\log(n+1)}\right]$$

$$= \lim_{n\to\infty} n \frac{\log\left(\frac{n}{n+1}\right)}{\log(n+1)}.$$

$$= \lim_{n\to\infty} \frac{\log\left(\frac{n}{n+1}\right)}{\left(\frac{\log(n+1)}{n}\right)} = \boxed{0 < 1}$$

↳ Divergent

$\therefore$ Convergent: $x < 1$

Divergent: $x \geq 1$.

Q. $\Sigma u_n = \left(\frac{1}{2}\right)^p + \left(\frac{1\cdot 3}{2\cdot 4}\right)^p + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^p + \ldots\ldots$

$$u_n = \left\{\frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots(2n)}\right\}^p$$

$$u_{n+1} = \left\{\frac{1\cdot 3\ldots(2n-1)(2n+1)}{2\cdot 4\ldots(2n)(2n+2)}\right\}^p$$

$$\frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^p.$$

HW. Evaluate the Raabe's Test: