31 October 2023 21:13

$u_1 \qquad u_2 \qquad u_3$

Q. $$\frac{2^2}{3\cdot 4}x^4 + \frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6}x^6 + \frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}x^8 + \dots\dots = \sum u_n$$

Series converges if: (a) $x^2 < 1$ (b) $x^2 \le 1$ (c) $x^2 > 1$ (d) $x^2 \ge 1$

$$u_n = \frac{\prod (2n)^2 \cdot x^{2n} \cdot x^2}{\prod (2n+1)(2n+2)} = \frac{2^2\cdot 4^2 \dots (2n)^2}{3\cdot 4 \dots (2n+1)(2n+2)} x^{2n+2}$$

D-Alembert's Test:

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} =$$

$$u_{n+1} = \frac{2^2\cdot 4^2 \dots (2n)^2 (2n+2)^2}{3\cdot 4 \dots (2n+1)(2n+2)(2n+3)(2n+4)} x^{2n+4}$$

$$= u_n \cdot \frac{(2n+2)^2}{(2n+3)(2n+4)} \cdot x^2$$

$$\therefore \lim_{n\to\infty} \frac{(2n+2)^2}{(2n+3)(2n+4)} x^2$$

$$= x^2 \implies$$

$x^2 < 1 \implies$ convergent

$x^2 > 1 \implies$ Divergent.

Check $x^2 = 1$; Proceed with Raabe's Test:-

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \lim_{n\to\infty} n\left[\frac{(2n+3)(2n+4)}{(2n+2)^2} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\cancel{4n^2}+14n+12-\cancel{4n^2}-8n-4}{(2n+2)^2}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{6n+8}{(2n+2)^2}\right] = \frac{6}{4} = \frac{3}{2} > 1$$

If $x^2 = 1$, By Raabe's Test, series is convergent.

$\therefore$ Combining: $x^2 \le 1 \implies$ series is convergent.

(5) Logarithmic Test:

Test.

Let $\sum u_n$ be a series of positive terms.

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = l \quad , \quad \begin{array}{l} l > 1 \Rightarrow \text{Series is convergent} \\ l < 1 \Rightarrow \text{Series is divergent} \end{array}$$

(6) Gauss Test:

Let $\sum u_n$ be a series of positive terms.

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma_n}{n^p}.$$

where $\alpha > 0$, $p > 1$ and $\{\gamma_n\}$ is a bounded seq.

(i) If $\alpha \neq 1$, $\sum u_n$ converges if $\alpha > 1$ & diverges if $\alpha < 1$.

(ii) If $\alpha = 1$, $\sum u_n$ converges if $\beta > 1$ & diverges if $\beta \leq 1$.

Q. $1 + \frac{x}{1!} + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \cdots\cdots , \quad x > 0.$

$$u_n = \frac{x^n \cdot n^n}{n!}$$

$$u_{n+1} = \frac{(x)^{n+1} \cdot (n+1)^{n+1}}{(n+1)!}$$

Logarithmic Test: $\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}}$.

$$\frac{u_n}{u_{n+1}} = \frac{\cancel{x^n} \cdot n^n}{\cancel{n!}} \times \frac{(n+1)\, \cancel{n!}}{x^{\cancel{n}+1} \cdot (n+1)^{n+1}} = \frac{n^n}{x\,(n+1)^n}.$$

$$= \frac{1}{x \cdot \left(1 + \frac{1}{n}\right)^n}.$$

$$\lim_{n\to\infty} n \log\left[\frac{1}{x\cdot\underbrace{\left(1+\frac{1}{n}\right)^n}_{e}}\right]$$ ---- (Logarithmic Test not directly applicable)

$$\frac{u_{n+1}}{u_n} = x\left(1+\frac{1}{n}\right)^n$$

D-Alembert's Ratio test: $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} x\left(1+\frac{1}{n}\right)^n$

$= x\cdot e$.

$x\cdot e < 1 \Rightarrow x < \frac{1}{e} \Rightarrow$ Convergent

$x\cdot e > 1 \Rightarrow x > \frac{1}{e} \Rightarrow$ Divergent.

For $x = \frac{1}{e}$, Check using Logarithmic Test:-

$$\frac{u_n}{u_{n+1}} = \frac{1}{x\left(1+\frac{1}{n}\right)^n} = \frac{e}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty} n \log\left[\frac{e}{\left(1+\frac{1}{n}\right)^n}\right]$$

$$= \lim_{n\to\infty} n\cdot\log e - n^2\log\left(1+\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} n - n^2 \log\left(1+\frac{1}{n}\right)$$ HW.

Note: $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\cdots$; $-1 < x \le 1$.

$n \ge 1 \Rightarrow \frac{1}{n} \le 1$.

$$\log\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2}\cdot\frac{1}{n^2} + \frac{1}{3}\cdot\frac{1}{n^3} - \cdots\cdots$$

$\log(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \cdots\cdots$, $-1 < -x \le 1$

$$\log(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \cdots\cdots, \quad -1 < -x \leq 1$$

Q. $x + x^{\left(1+\frac{1}{2}\right)} + x^{\left(1+\frac{1}{2}+\frac{1}{3}\right)} + \cdots\cdots$ is :-

(a) Convergent if $x \geq \frac{1}{e}$

(b) Divergent if $x \geq \frac{1}{e}$

(c) Convergent if $x > \frac{1}{e}$

(d) Divergent if $x > \frac{1}{e}$

$$\left.\begin{aligned} u_n &= x^{\Sigma\left(\frac{1}{n}\right)} \\ u_{n+1} &= x^{\Sigma\left(\frac{1}{n}\right) + \frac{1}{n+1}} \end{aligned}\right\} \Rightarrow \frac{u_n}{u_{n+1}} = \frac{1}{x^{1/n+1}}.$$

Logarithmic Test: $\lim\limits_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \lim\limits_{n\to\infty} n \log\left(\frac{1}{x^{1/n+1}}\right)$

$$= \lim_{n\to\infty} n \cdot \log\left(\frac{1}{x}\right)^{\frac{1}{n+1}}$$

$$= \lim_{n\to\infty} \frac{n}{n+1} \cdot \log\left(\frac{1}{x}\right)$$

$$= \log\left(\frac{1}{x}\right).$$

$\log\left(\frac{1}{x}\right) > 1 \Rightarrow \frac{1}{x} > e \Rightarrow x < \frac{1}{e} \Rightarrow$ Convergent

$x > \frac{1}{e} \Rightarrow$ Divergent.

When $x = \frac{1}{e}$: $\frac{u_n}{u_{n+1}} = \frac{1}{x^{\frac{1}{n+1}}} = \frac{1}{\left(\frac{1}{e}\right)^{\frac{1}{n+1}}}.$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \left(\frac{1}{e}\right)^{\frac{1}{n+1}} \to 1$$

Raabe's Test: $\lim\limits_{n} \left[\frac{u_n}{\cdots} \cdots\right] \cdots \left[\frac{1}{\cdots} \cdots\right]$

Raabe's Test: $\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \lim_{n\to\infty} n\left[\frac{1}{\left(\frac{1}{e}\right)^{\frac{1}{n+1}}} - 1\right]$ ... HW