PRF : $Y_i = \alpha + \beta X_i + u_i$

SRF : $\hat{Y}_i = \hat{\alpha} + \hat{\beta} X_i$

Using OLS : $\hat{\beta} = \dfrac{\sum y_i x_i}{\sum x_i^2}$, $\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$

$y_i = (Y_i - \bar{Y})$

$x_i = (X_i - \bar{X})$

$\therefore \; Var(\hat{\beta}) = \dfrac{\sigma^2}{\sum x_i^2}$, $Var(\hat{\alpha}) = \dfrac{\sigma^2 \sum X_i^2}{n \sum x_i^2}$

$\therefore \; \widehat{Var(\hat{\beta})} = \dfrac{\hat{\sigma}^2}{\sum x_i^2}$, $\widehat{Var(\hat{\alpha})} = \dfrac{\hat{\sigma}^2 \sum X_i^2}{n \sum x_i^2}$, $\hat{\sigma}^2 = \left(\dfrac{\sum e_i^2}{n-2}\right)$

$s.e(\hat{\beta}) = \sqrt{\widehat{Var(\hat{\beta})}}$, $s.e(\hat{\alpha}) = \sqrt{\widehat{Var(\hat{\alpha})}}$

$R^2 = \dfrac{ESS}{TSS}$.

Q. Consider the following data:

$\sum X_i = 20$, $\sum Y_i = 40$, $\sum (X_i - \bar{X})^2 = 40$, $\sum (Y_i - \bar{Y})^2 = 124$

$\sum (X_i - \bar{X})(Y_i - \bar{Y}) = 70$, $n = 5$

(i) Estimate the parameters α, β for the model: $Y_i = \alpha + \beta X_i + u_i$.

$\hat{\beta} = \dfrac{\sum y_i x_i}{\sum x_i^2} = \dfrac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} = \dfrac{70}{40} = \dfrac{7}{4} = 1.75$

$\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$, Here $\bar{Y} = \dfrac{\sum Y_i}{n} = \dfrac{40}{5} = 8$, $\bar{X} = \dfrac{\sum X_i}{n} = \dfrac{20}{5} = 4$.

$\hat{\alpha} = 8 - \left(\dfrac{7}{\cancel{4}}\right) \times \cancel{4}$

$\hat{\alpha} = 8 - 7 = 1$.

(ii) Find the s.e of the estimates:

$\therefore \widehat{Var(\hat\beta)} = \dfrac{\hat\sigma^2}{\boxed{\sum x_i^2}}$, $\widehat{Var(\hat\alpha)} = \dfrac{\hat\sigma^2 \boxed{\sum X_i^2}}{\boxed{n}\boxed{\sum x_i^2}}$, $\hat\sigma^2 = \left(\dfrac{\sum e_i^2}{\boxed{n-2}}\right)$ RSS.

$\sum x_i^2 = \boxed{\sum (X_i - \bar X)^2 = \sum X_i^2 - n\bar X^2}$ ---- $(*)$

$Var(X) = \frac{1}{n}\sum (X_i - \bar X)^2 = \boxed{\frac{1}{n}\sum X_i^2 - \bar X^2}$

$\sum X_i^2 = \sum (X_i - \bar X)^2 + n\bar X^2 = 40 + 5(4)^2$

$= 40 + 5 \times 16$

$= 40 + 80 = 120.$

From ANOVA, $TSS = ESS + \boxed{RSS}$ (solve for!)

$= 124$

$ESS = \sum (\hat Y_i - \bar{\hat Y})^2 = \hat\beta^2 \sum (X_i - \bar X)^2 = \left(\frac{7}{4}\right)^2 \times 40 = \frac{49}{\cancel{16}} \times \cancel{40} = \frac{245}{2}$

$= 122.5$

$\therefore RSS = TSS - ESS = 124 - 122.5 = 1.5$

$\therefore \hat\sigma^2 = \dfrac{RSS}{n-2} = \dfrac{1.5}{5-2} = \dfrac{1.5}{3} = \dfrac{3/2}{3} = \dfrac{1}{2} = 0.5$

$\therefore \widehat{Var(\hat\beta)} = \dfrac{\hat\sigma^2}{\sum x_i^2} = \dfrac{1/2}{40} = \dfrac{1}{80}.$

$\widehat{Var(\hat\alpha)} = \dfrac{\hat\sigma^2 \sum X_i^2}{n\sum x_i^2} = \dfrac{\frac{1}{2} \times 120}{5 \times 40} = \dfrac{\cancel{60}}{5 \times \cancel{40}} = \dfrac{3}{10}.$

$\therefore s.e(\hat\beta) = \sqrt{\dfrac{1}{80}} = \dfrac{1}{4\sqrt5}$, $s.e(\hat\alpha) = \sqrt{\dfrac{3}{10}}.$

$= 0.11$ $\qquad$ $= 0.54.$

Q. $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Use MLE to find μ & σ.

$X_1 \sim N(\mu, \sigma^2) \Rightarrow f(x_1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_1-\mu}{\sigma}\right)^2}$

$X_2 \sim N(\mu, \sigma^2)$

$\Rightarrow f(x_2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_2-\mu}{\sigma}\right)^2}$

$\therefore$ Joint distn $= \prod_{i=1}^{n} f(x_i) = f(x_1).f(x_2)\ldots f(x_n)$

$$= \left\{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_1-\mu}{\sigma}\right)^2}\right\}\left\{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_2-\mu}{\sigma}\right)^2}\right\}\ldots\left\{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_n-\mu}{\sigma}\right)^2}\right\}$$

$$= \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2} = \text{Likelihood fn.}$$

(1) $L(\mu, \sigma^2) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}$

(2) Log-likelihood $\ell(\mu, \sigma^2) = n\log\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$

$$= -n\log(\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$$

(3). For MLE:

$\frac{\partial \ell}{\partial \mu} = 0 \Rightarrow -\frac{1}{\cancel{2}\sigma^2}(\cancel{2})\sum(x_i-\mu) = 0. \Rightarrow \sum(x_i-\mu) = 0$

$\Rightarrow \sum x_i = n\mu \Rightarrow \boxed{\hat{\mu}_{MLE} = \bar{x}}$

$\frac{\partial \ell}{\partial \sigma} = 0 \Rightarrow -n.\frac{1}{\sigma\cancel{\sqrt{2\pi}}}\cancel{\sqrt{2\pi}} - \frac{1}{\cancel{2}}.\left(-\frac{\cancel{2}}{\sigma^3}\right).\sum(x_i-\mu)^2 = 0.$

$\Rightarrow -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum(x_i-\mu)^2 = 0$

$\Rightarrow \frac{n}{\cancel{\sigma}} = \frac{1}{\sigma^{\cancel{3}}}\sum(x_i-\mu)^2$

$$\Rightarrow \boxed{\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum(x_i - \mu)^2}$$

Q. Consider the model: $Y_i = \beta X_i + u_i$ [Model without intercept].

(i) Using OLS find the estimate of β and show that it is unbiased.

PRF : $Y_i = \beta X_i + u_i$

SRF : $\hat{Y}_i = \hat{\beta} X_i$

$\therefore$ For OLS, Min $\sum e_i^2 = \sum(Y_i - \hat{Y}_i)^2 = \sum(Y_i - \hat{\beta}X_i)^2$.

$$\frac{\partial \sum e_i^2}{\partial \hat{\beta}} = 0 \Rightarrow (-2)\sum(Y_i - \hat{\beta}X_i)X_i = 0.$$

$$\Rightarrow \sum Y_i X_i = \hat{\beta}\sum X_i^2 \Rightarrow \boxed{\hat{\beta} = \frac{\sum Y_i X_i}{\sum X_i^2}}$$

Checking for unbiaseness of $\hat{\beta}$:-

$$\hat{\beta} = \frac{\sum(\beta X_i + u_i)X_i}{\sum X_i^2} = \frac{\beta\sum X_i^2 + \sum u_i X_i}{\sum X_i^2}$$

$$\hat{\beta} = \beta + \frac{\sum u_i X_i}{\sum X_i^2}$$

$$E(\hat{\beta}) = \beta + \frac{\sum \underbrace{E(u_i)}_{=0} \cdot X_i}{\sum X_i^2} = \beta.$$

[$\therefore \hat{\beta}$ is an unbiased estimator of β]