DIFFERENTIAL CALCULUS 14.4.24

11 April 2024 ~~10:20~~

15% (I) Limits

30% (II) Function question

25% (III) Advanced level Cont/Diff

30% (IV) Differential equation

#

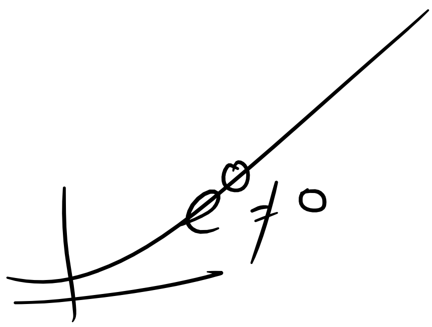
Differential Calculus

A

1. Let $f: R^2 \to R^2$ be defined by $f(x, y) = (e^x \cos(y), e^x \sin(y))$. Then, the number of points in R^2 that do not lie in the range of f is

(a) 0 (b) 1 (c) 2 (d) infinite

n dim Real



$$f(x,y) = (e^x \cos y, e^x \sin y)$$

$$f(z) = e^z = e^{x+iy} = e^x(\cos y + i \sin y) = (e^x \cos y, e^x \sin y)$$

$$= e^x$$

$\boxed{f(z) = e^z}$ → lie all pts in R^2 except $(0,0)$

1 point

finally, $\rightarrow 4e^2\left[1+2y^2+2x^2+\cancel{4x^2y^2}-\cancel{4x^2y^2}\right]$

$= 4e^2\left[1+2(x^2+y^2)\right]$

$= 4e^2\left[1+2\right] = \underline{12e^2}$

$4e^2\begin{vmatrix} 1+2x^2 & 2xy \\ 2xy & 1+2y^2 \end{vmatrix}$

2 6. Let $f(x,y)=e^{x^2+y^2}$ for $(x,y)\in R^2$, and a_n be the determinant of the matrix $\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y} \\ \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$ evaluated at the point $(\cos n, \sin n)$. Then, the limit $\lim_{n\to\infty} a_n$ is

(a) non-existent (b) 0 (c) $6e^2$ (d) $12e^2$

$f_x = e^{x^2+y^2}(2x)$ $\quad f_y = e^{x^2+y^2}(2y)$

$f_{xx} = e^{x^2+y^2}(2x)^2 + e^{x^2+y^2}\cdot 2$

$= 2e^{x^2+y^2}(1+2x^2)$

$f_{yy} = 2e^{x^2+y^2}(1+2y^2)$ $\quad x^2+y^2$

$f_{xy} = f_{yx} = 4xye^{x^2+y^2}$

@ $\boxed{f_{yy} = 2e(1+2y^2)}$ $= 2e(1+2x^2)$

$f_{xy} = f_{yx} = 4xye$

L min

dift

$f_{xy} = f_{yx}$ $\quad \boxed{Q = R}$ $\quad QR = Q^2$

2 7. Let $f(x,y)=\ln(1+x^2+y^2)$ for $(x,y)\in R^2$. Define

$P = \frac{\partial^2 f}{\partial x^2}\Big|_{(0,0)}$, $Q = \frac{\partial^2 f}{\partial x\,\partial y}\Big|_{(0,0)}$ $\Rightarrow$ $R = \frac{\partial^2 f}{\partial y\,\partial x}\Big|_{(0,0)}$, $S = \frac{\partial^2 f}{\partial y^2}\Big|_{(0,0)}$

Then,

(a) $PS-QR>0$ and $P<0$

(b) $PS-QR>0$ and $P>0$

(c) $PS-QR<0$ and $P>0$

(d) $PS-QR<0$ and $P<0$

local minima

(a) $PS - QR > 0$ and $P < 0$
(b) $PS - QR > 0$ and $P > 0$
(c) $PS - QR < 0$ and $P > 0$
(d) $PS - QR < 0$ and $P < 0$

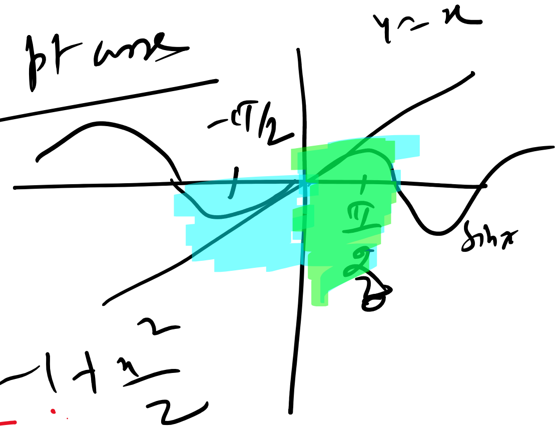
$f(0,0) = 0$ $\quad f(x,y) \geq f(0,0) \rightarrow$ local minima

$f_{xx} \cdot f_{yy} - (f_{xy})^2 > 0 \quad f_{xx} > 0$

$PS - Q^2 > 0$

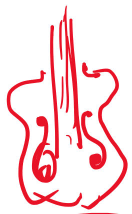
$PS - QR > 0 \quad P > 0$

Functional curve Analysis   don't indicate pt ...



25. Let $f(x) = \cos x$ and $g(x) = 1 - \frac{x^2}{2}$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then,

(a) $f(x) \geq g(x), \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(b) $f(x) \leq g(x), \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(c) $f(x) - g(x)$ changes sign exactly once on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(d) $f(x) - g(x)$ changes sign more than once on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



Coal Blast Box

$h(x) = f(x) - g(x) = \cos x - 1 + \frac{x^2}{2}$

$h'(x) = -\sin x + x$

Graphically $h'(x)$ is lying $(-\pi/2, 0)$

$h(0) = 0$

↑ in $(0, \pi/2)$

$h(x) \geq 0 \ \forall\ x \in (-\pi/2, \pi/2)$

$f(x) \geq g(x) \ \forall\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$f(x) > g(x)$

6

6. Let $y: R \to R$ be a twice differentiable function such that y'' is continuous on $[0, 1]$ and $y(0) = y(1) = 0$. Suppose, $y''(x) + x^2 < 0 \; \forall \; x \in [0, 1]$. Then,

(a) $y(x) > 0 \; \forall \; x \in (0, 1)$
(b) $y(x) < 0 \; \forall \; x \in (0, 1)$
(c) $y(x) = 0$ has exactly one solution in $(0, 1)$
(d) $y(x) = 0$ has more than one solution in $(0, 1)$

$x = 0 \quad x = 1$

Most easy

Let $y(x) = x(1-x)$

$y(0) = 0$
$y(1) = 0$

$y'(x) = -2x + 1$
$\Rightarrow y''(x) = -2$

$y''(x) + x^2 = -2 + x^2$

$0 < x < 1$

$0 \le x^2 \le 1$

$-2 + 0 \le +x^2 - 2 \le -2 + 1$

$-2 \le y''(x) + x^2 \le -1$

So, $y''(x) + x^2 < 0$ is also true

$0 < x < 1$

$0 > -x > -1$

$0 < 1 - x < 1$

$x(1-x) > 0$

$y(x) > 0$

$y = e^x$

$f(x) \to$ Biquadratic

7. Let $f: R \to R$ be an infinitely differentiable function such that f'' has exactly two distinct zeroes. Then

(a) f' has atmost 3 distinct zeroes
(b) f' has atleast 1 zero
(c) f has atmost 3 distinct zeroes
(d) f has atleast 2 distinct zeroes

4

2 zeros $\to$ Quadratic

$f''(x) = ax^2 + bx + ce^x$

$ax^2 - bx + ce^{-x}$

(4) 2 ≥ cross → ...

$f''(x) = ax^2 + bx + c$

Let, $f''(-x) = ax^2 - bx + c$

+ve	0	0
−ve	2	0
img	0	2

$f'(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + ce^x + d$

$f'(-x) = -\frac{ax^3}{3} + \frac{bx^2}{2} + ce^{-x} + d$

$f(x) = \frac{ax^4}{12} + \frac{bx^3}{6} + ce^x + dx + e$

$f(-x) = \frac{ax^4}{12} - \frac{bx^3}{6} + ce^{-x} - dx + e$

−ve Real Roots

+ve → 0
−ve → 1
Img → 2

At least 1 Real Root

$|f'(x) - 0| < M|x - 0|$

$|f'(x) - f'(0)| \le M(x - 0)$

$f'(x)$ → is cont at $x = 0$

Hint

8. Let $f:(-1, 1) \to R$ be a differentiable function satisfying $f(0) = 0$. Suppose, there exists an $M > 0$, such that $|f'(x)| \le M|x| \ \forall\ x \in (-1, 1)$. Then,

(a) f' is continuous at $x = 0$
(b) f' is differentiable at $x = 0$
(c) ff' is differentiable at $x = 0$
(d) $(f')^2$ is differentiable at $x = 0$

Now, let, $f(x) = x|x| = \begin{cases} x^2 & x > 0 \\ -x^2 & x \le 0 \end{cases}$

$f(0) = 0$

$f'(x) = 2|x|$

Check differentiability not diff at $x = 0$

$f(t,t) = \frac{t^7}{2t^6} - 0$ Then not homogeneous

$\frac{\lambda^3(\lambda x^4 + y^3)}{\lambda^6(x^6+y^6)}$ $\frac{x^4+y^3}{x^6+y^6} \rightarrow \frac{\lambda^4 \lambda^3 \cdot x^4 y^3}{\lambda^6(x^6+y^6)}$

$\neq \frac{1}{2}$

9. Let $f: R^2 \rightarrow R$ be defined as follows :

$$f(x,y) = \begin{cases} \frac{x^4 y^3}{x^6+y^6}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

→ Homogeneous of degree 1

$f(0,k) = \frac{0^4 \cdot k^3}{0^6+k^6} = 0$

$= \frac{\lambda^7(x^4y^3)}{\lambda^6(x^6+y^6)} = \lambda^1 f(x,y)$

Then,

(a) $\lim_{t \to 0} \frac{f(t,t) - f(0,0)}{t}$ exists and equals $\frac{1}{2}$

(b) $\left.\frac{\partial f}{\partial x}\right|_{(0,0)}$ exists and equals 0

(c) $\left.\frac{\partial f}{\partial y}\right|_{(0,0)}$ exists and equals 0

(d) $\lim_{t \to 0} \frac{f(t,2t) - f(0,0)}{t}$ exists and equals $\frac{1}{3}$ X

$\frac{8}{65}$ lt ?

d $\quad \lim_{t \to 0} \frac{\frac{t^4 \cdot 8t^3}{t^6 + 64t^6} - 0}{t} \rightarrow$

$\lim_{k \to 0} \frac{0-0}{k} = 0$

b $\quad \left.\frac{\partial f}{\partial y}\right|_{x,y} = \lim_{k \to 0} \frac{f(x, y+k) - f(0,0)}{k} = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$

Also true c

B

A

L ??

10. The value of $\lim_{n \to \infty} \left(n \int_0^1 \frac{x^n}{x+1} dx \right)$ is equal to .... (rounded off to two decimal places)

$I_n = \int_0^1 \frac{x^n}{x+1} dx$

$\int^1 x^{n-1} \left(\frac{1-e^{n}-1}{1-e^{n}} \right) dx$

$$I_n = \int_0^1 \frac{x^n}{x+1}\,dx = \int_0^1 x^{n-1}\left(\frac{1+x}{1+x}\right)\ldots$$

$$= \int_0^1 x^{n-1}\,\frac{x\,dx}{1+x} = \int_0^1 x^{n-1}\left(1 - \frac{1}{1+x}\right)dx$$

$$= \int_0^1 x^{n-1} \ldots$$

$$I_n = \left[\frac{x^n}{n}\right]_0^1 - I_{n-1}$$

$$\boxed{I_n = \frac{1}{n} - I_{n-1}}$$

When $\lim_{n\to\infty}$ $\boxed{I_{n-1} = I_n}$

$$2I_n = \frac{1}{n}$$

$$I_n = \frac{1}{2n}$$

$$\lim_{n\to\infty} n\left(\frac{1}{2n}\right) = 0.5$$

52. The global minimum value of $f(x) = |x-1| + |x-2|^2$ on R is equal to .... (rounded off to two decimal places)