16 August 2023 12:14

F. Given that

$$\sin(5\theta) = 5\sin\theta - 20(\sin\theta)^3 + 16(\sin\theta)^5$$

for all real θ, it follows that the value of $\sin(72^\circ)$ is

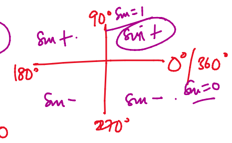
(a) $\sqrt{\frac{5+\sqrt{5}}{8}}$, (b) 0, (c) $-\sqrt{\frac{5+\sqrt{5}}{8}}$,

(d) $\sqrt{\frac{5-\sqrt{5}}{8}}$, (e) $-\sqrt{\frac{5-\sqrt{5}}{8}}$.

$\frac{7.2}{8}$. $\frac{2.8}{8}$.

$\sin 30 = \frac{1}{2}$

$\sin 90 = 1$



$\theta = 72^\circ$ $5\theta = 360^\circ$ $\sin\theta = x$. $\sin 5\theta = 0$

$\sin 60^\circ < \sin 72^\circ < 1$ $\boxed{0.85 < \sin 72^\circ < 1}$ $x \neq 0$

$\frac{\sqrt{3}}{2} = \frac{1.7}{2} = 0.85$

$5x - 20x^3 + 16x^5 = 0$.

$16x^4 - 20x^2 + 5 = 0$

$\sqrt{5} = 2.2$

$x^2 = \frac{20 \pm \sqrt{400 - 4\times16\times5}}{2\times16} = \frac{20 \pm \sqrt{80}}{32} = \frac{20 \pm 4\sqrt{5}}{32}$

$0.7225 < \sin^2 72^\circ < 1$

$= \frac{5 \pm \sqrt{5}}{8} = \frac{5+2.2}{8}, \frac{5-2.2}{8} = \frac{7.2}{8}, \frac{2.8}{8}$

$x^2 = \frac{5+\sqrt{5}}{8}$ $= 0.9,\ 0.35$

$\sqrt{11.2} = 3 + \frac{2.2}{2\times3} \sqrt{5} = 2 + \frac{1}{2\times2} = 2 + \frac{1}{4} = \boxed{2.25}$ $\boxed{2.2}36$

9 $= 3 + 0.3$ $\sqrt{4} = 2$

$= 3.3$.

$\sqrt{80} = 8 + \frac{16}{2\times8} = 8 + 1 = 9$, $\sqrt{80} = 8.9$.

64

$\sqrt{39} = (36+3)^{1/2} = 36^{1/2} + \frac{1}{2}\times(36)^{1/2 - 1} \times 3.$

$(a+y)^n = x^n + nx^{n-1}y + \ldots$ $= 6 + \frac{1}{2}\times\frac{1}{6_2}\times 3 = 6 + \frac{1}{4}$

$= 6.2$.

G. For all real n, it is the case that $n^4 + 1 = (n^2 + \sqrt{2}n + 1)(n^2 - \sqrt{2}n + 1)$. From this we may deduce that $n^4 + 4$ is

$a^2 - b^2 = (a+b)(a-b)$

(a) never a prime number for any positive whole number n. $0, 1, 2 \ldots$

(b) a prime number for exactly one positive whole number n. $= 1$

(c) a prime number for exactly two positive whole numbers n.

(d) a prime number for exactly three positive whole numbers n.

(e) a prime number for exactly four positive whole numbers n.

$n^4 + 1 = (n^4 + 1 + 2n^2) - 2n^2$

$= (n^2+1)^2 - (\sqrt{2}n)^2$

$n = 1$

$n^4 + 4 = 5$

$= (n^2 + \sqrt{2}n + 1)(n^2 - \sqrt{2}n + 1)$

$\sqrt{2}n = m \rightarrow n = \frac{m}{\sqrt{2}}$. $n^4 = \frac{m^4}{4}$ $n^2 = \frac{m^2}{2}$

any prime no (except 2 & 3) can be expressed as $6n \pm 1$

$\frac{m^4}{4} + 1 = \left(\frac{m^2}{2} + m + 1\right)\left(\frac{m^2}{2} - m + 1\right)$

$m^4 + 4 = (m^2 + 2m + 2)(m^2 - 2m + 2)$

$(m^2)^2 + (2)^2 + 2 \cdot m^2 \cdot 2$ $m^4 + 4 = (m^4 + 4 + 4m^2) - 4m^2 = (m^2+2)^2 - (2m)^2 = (m^2+2m+2)(m^2-2m+2)$

prime $\leftarrow$ $n^4 + 4 = (n^2 + 2n + 2)(n^2 - 2n + 2)$

$p = 1 \times p$. $p = 1 \times p$

$n^2 \pm 2n + 2 = 1$

$n^2 \pm 2n + 1 = 0$. $n = -1, 1$

$\boxed{x^2 + x + 1} = \left[x^2 + 2\times\frac{1}{2}\times x + \left(\frac{1}{2}\right)^2\right] - \left(\frac{1}{2}\right)^2 + 1$ $R^2 \geq 0$

k. k^2 k^2 $(R^2)_{min} = 0$

form $(x+k)^2 \pm m^2$. $2k = 1$

$x^2 + 2kx + k^2 \pm m^2$. $k = \frac{1}{2}$

$= \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + 1$ $= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$.

min $= 0$

$4 - (x+3)^2$.

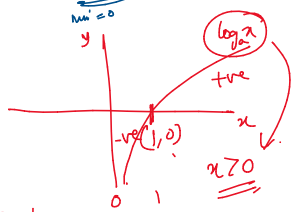
min $= 0$

H. How many real solutions x are there to the following equation?

$$\log_2(2x^3 + 7x^2 + 2x + 3) = 3\log_2(x+1) + 1$$

>0 >0

(a) 0, (b) 1, (c) 2, (d) 3, (e) 4.



$x > -1$

$x > 0$

$\log a + \log b = \log(ab)$ $x\log y = \log(y^x)$

$3\log_2(x+1) = \log_2(x+1)^3$ $\log_a a = 1$

$\log_2 2 = 1$

$a^0 = 1$ except for $a = 0$

$\log_2(2x^3 + 7x^2 + 2x + 3) = \log_2(x+1)^3 + \log_2 2$

$0^{1-1} = \frac{0^1}{0^1} = 0/0$. $a^{(1-1)} = \frac{a^1}{a^1} = \frac{a}{a} = 1$

$= \log_2[2(x+1)^3]$

$(a+b)^3 =$

$\cancel{2x^3} + 7x^2 + 2x + 3 = 2(x^3 + 3x^2 + 3x + 1)$

$= \cancel{2x^3} + 6x^2 + 6x + 2$

$x^2 - 4x + 1 = 0.$

$x^2 - 4x + 4 - 4 + 1 = 0$

$(x-2)^2 = 3.$

$x - 2 = \pm\sqrt{3}$

$x = 2 \pm \sqrt{3} = 2+\sqrt{3}, \ 2-\sqrt{3}$

+ve +ve

$\sqrt{3} = 1.7$

I. Alice and Bob each toss five fair coins (each coin lands on either heads or tails, with equal probability and with each outcome independent of each other). Alice wins if strictly more of her coins land on heads than Bob's coins do, and we call the probability of this event p_1. The game is a draw if the same number of coins land on heads for each of Alice and Bob, and we call the probability of this event p_2. Which of the following is correct?

(a) $p_1 = \frac{193}{512}$ and $p_2 = \frac{63}{256}$.

(b) $p_1 = \frac{201}{512}$ and $p_2 = \frac{55}{256}$.

(c) $p_1 = \frac{243}{512}$ and $p_2 = \frac{13}{256}$.

(d) $p_1 = \frac{247}{512}$ and $p_2 = \frac{9}{256}$.

(e) $p_1 = \frac{1}{3}$ and $p_2 = \frac{1}{3}$.

$P(H) = \frac{1}{2} \quad P(T) = \frac{1}{2}$

Toss 1 Toss 2 Toss 3 Toss 4 Toss 5

$P(3H) \rightarrow$ any 3 of 5

${}^5C_3 \times \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^2$

$p(s) = x$

$p(f) = (1-x)$

A	B
0	0
1	1
2	2
3	3
4	4
5	5

${}^5C_0\left(\frac{1}{2}\right)^5 \, {}^5C_0\left(\frac{1}{2}\right)^5.$

${}^5C_0^{\,2} \cdot \frac{1}{2^{10}} = \frac{1}{2^{10}}$

${}^5C_1^{\,2} \cdot \frac{1}{2^{10}} = \frac{25}{2^{10}}$

${}^5C_2^{\,2} \cdot \frac{1}{2^{10}} = \frac{100}{2^{10}}$

${}^5C_3^{\,2} \cdot \frac{1}{2^{10}} = \frac{100}{2^{10}}$

${}^5C_4^{\,2} \cdot \frac{1}{2^{10}} = \frac{25}{2^{10}}$

${}^5C_5^{\,2} \cdot \frac{1}{2^{10}} = \frac{1}{2^{10}}$

$p_1 = \frac{1}{2}\left[1 - \frac{63}{256}\right]$

${}^5C_4 = {}^5C_1 = 5$

${}^5C_3 = {}^5C_2 = 10$

$p_2 = \frac{252}{1024} = \frac{63}{256}$

P1

	A	B
(a)	5	4, 3, 2, 1, 0
(b)	4	3, 2, 1, 0
(c)	3	2, 1, 0
(d)	2	1, 0
(e)	1	0

(a) ${}^5C_5\left(\frac{1}{2}\right)^5\left[{}^5C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) + {}^5C_3\left(\frac{1}{2}\right)^5 + {}^5C_2\left(\frac{1}{2}\right)^5 + {}^5C_1\left(\frac{1}{2}\right)^5 + {}^5C_0\left(\frac{1}{2}\right)^5\right]$

$= \frac{1}{2^{10}}\left[5 + 10 + 10 + 5 + 1\right] = \frac{31}{2^{10}}.$

(b) ${}^5C_4 \frac{1}{2^{10}}\left[{}^5C_3 + {}^5C_2 + {}^5C_1 + {}^5C_0\right] = \frac{5}{2^{10}}\left[10+10+5+1\right] = \frac{130}{2^{10}}.$

(c) ${}^5C_3 \frac{1}{2^{10}}\left[{}^5C_2 + {}^5C_1 + {}^5C_0\right] = \frac{10}{2^{10}}\left[10+5+1\right] = \frac{160}{2^{10}}.$

(d) ${}^5C_2 \frac{1}{2^{10}}\left[{}^5C_1 + {}^5C_0\right] = \frac{10}{2^{10}}(5+1) = \frac{60}{2^{10}}.$

(e) ${}^5C_1 \frac{1}{2^{10}} \cdot {}^5C_0 = \frac{5}{2^{10}}$

$\frac{386}{2^{10}} = \frac{386}{1024} = \frac{193}{512}$

$P(A) = \frac{193}{512} \quad P(B) = \frac{193}{512}$

$P(\text{Draw}) = \frac{512-386}{512} = \frac{126}{512} = \frac{63}{256}. \quad \frac{386}{512}$

J. The real numbers m and c are such that the equation

$$x^2 + (mx+c)^2 = 1$$

has a repeated root x, and also the equation

$$(x-3)^2 + (mx+c-1)^2 = 1$$

has a repeated root x (which is not necessarily the same value of x as the root of the first equation). How many possibilities are there for the line $y = mx + c$?

(a) 0, (b) 1, (c) 2, (d) 3, (e) 4.

15 Oct

Computer Science

Recommended Books

(i) Suppose x, y, and z are whole numbers such that $x^2 - 19y^2 = z$. Show that for any such x, y and z, it is true that

$$(x^2 + Ny^2)^2 - 19(2xy)^2 = z^2$$

where N is a particular whole number which you should determine.

(ii) Find z if $x = 13$ and $y = 3$. Hence find a pair of whole numbers (x, y) with $x^2 - 19y^2 = 4$ and with $x > 2$.

(iii) Hence find a pair of positive whole numbers (x, y) with $x^2 - 19y^2 = 1$ and with $x > 1$.

Is your solution the only such pair of positive whole numbers (x, y)? Justify your answer.

(iv) Prove that there are no whole number solutions (x, y) to $x^2 - 25y^2 = 1$ with $x > 1$.

(v) Find a pair of positive whole numbers (x, y) with $x^2 - 17y^2 = 1$ and with $x > 1$.

(i) Sketch $y = (x^2 - 1)^n$ for $n = 2$ and for $n = 3$ on the same axes, labelling any points that lie on both curves, or that lie on either the x-axis or the y-axis.

(ii) Without calculating the integral explicitly, explain why there is no positive value of a such that $\displaystyle\int_0^a (x^2 - 1)^n \, dx = 0$ if n is even.

If $n > 0$ is odd we will write $n = 2m - 1$ and define $a_m > 0$ to be the positive real number that satisfies

$$\int_0^{a_m} (x^2 - 1)^{2m-1} \, dx = 0,$$

if such a number exists.

(iii) Explain why such a number a_m exists for each whole number $m \geqslant 1$.

(iv) Find a_1.

(v) Prove that $\sqrt{2} < a_2 < \sqrt{3}$.

(vi) Without calculating further integrals, find the approximate value of a_m when m is a very large positive whole number. You may use without proof the fact that $\displaystyle\int_0^{\sqrt{2}} (x^2 - 1)^{2m-1} \, dx < 0$ for any sufficiently large whole number m.