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Variance of Normal Distribution :

For central moments of even order,

$$\mu_{2r} = E(X-\mu)^{2r}$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2r} f(x)\,dx$$

$\bar{x} = \frac{1}{n}\sum x_i$

$= E(x)$

$\sigma^2 = \frac{1}{n}\sum x_i^2 - \bar{x}$

$E(x^2) - E(x)^2$

$\mu_r = \frac{1}{n}\sum(x_i - \bar{x})$

$$\mu_{2r} = \int_{-\infty}^{\infty} (x-\mu)^{2r} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let, $t = \frac{x-\mu}{\sigma}$

$\sigma\,dt = dx$

$$\mu_{2r} = \frac{1}{\sigma\sqrt{2\pi}}\int (\sigma t)^{2r} e^{-t^2/2}\, \cancel{\sigma}\, dt$$

$$= \frac{2\sigma^{2r}}{\sqrt{2\pi}} \int_0^{\infty} t^{2r} e^{-t^2/2}\, dt$$

Let $y = t^2/2$

$dy = \cancel{2}t\,dt / \cancel{2}$

$dt = \frac{dy}{t}$

$dt = \frac{dy}{\sqrt{2y}}$

$$= \frac{2\sigma^{2r}}{\sqrt{2\pi}} \int_0^{\infty} \left[\sqrt{2y}\right]^{2r} e^{-y} \frac{dy}{\sqrt{2y}}$$

$$= \frac{\cancel{2}\sigma^{2r}}{\cancel{2}\sqrt{\pi}} \int_0^{\infty} 2^{\frac{1}{2}\cdot 2r} (\sqrt{y})^{2r} e^{-y} \frac{dy}{\sqrt{y}}$$

$$= \frac{2^r \sigma^{2r}}{\sqrt{\pi}} \int_0^{\infty} y^{r-1/2} e^{-y}\, dy$$

$$\sqrt{\pi}$$

$$= \frac{2^r \sigma^{2r}}{\sqrt{\pi}} \int_0^\infty y^{r+1/2-1} e^{-y}\, dy$$

$$\int_0^\infty y^{-1/2} e^{-y}$$

$$\int_0^\infty y^{1/2-1} e^{-y}$$

$$\Gamma(1/2)$$

$$= \frac{2^r \sigma^{2r}}{\sqrt{\pi}} \Gamma(r+1/2)$$

$$= \frac{2^r \sigma^{2r}}{\sqrt{\pi}} \left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\left(r-\frac{5}{2}\right) \cdots \frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(1/2)$$

$$\mu_{2r} = \frac{2^r \sigma^{2r}}{\cancel{\sqrt{\pi}}} \left(\frac{2r-1}{2}\right)\left(\frac{2r-3}{2}\right)\left(\frac{2r-5}{2}\right) \cdots \frac{3}{2}\cdot\frac{1}{2}\cancel{\sqrt{\pi}}$$

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

$$\Gamma(5/2)$$

$$\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)$$

$$= \frac{\cancel{2^r}\sigma^{2r}}{\cancel{2^r}} \cdot (2r-1)(2r-3)\cdots 3\cdot 2\cdot 1$$

$$\therefore \boxed{\mu_{2r} = \sigma^{2r}\cdot(2r-1)(2r-3)\cdots 3\cdot 2\cdot 1}$$

In particular to calculate the variance

we put $r = 1$

that is $\mu_2 = \sigma^{2\cdot 1}\cdot 1$

$\mu_2 = \sigma^2 =$ variance (proved)

Standard Normal Distribution

$X \sim N(\mu, \sigma^2)$

Linear conversion

$y = a + bx$

$x = \mu + \sigma t$

$t = \frac{x-\mu}{\sigma}$

$$t = \frac{x-\mu}{\sigma}$$

Let $t = \frac{x-\mu}{\sigma}$ where x is a normal variable with mean 'μ' and std deviation σ.

$$E(t) = \frac{E(x)-\mu}{\sigma} = \frac{\mu-\mu}{\sigma} = 0$$

and $\mathrm{var}(t) = v\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma^2} v(x) = \frac{\sigma^2}{\sigma^2} = 1$

$t \sim N(0,1) \Rightarrow$ this is a standard normal variate.

and we can write down the p.d.f of t by

$$\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \quad -\infty < t < \infty$$

$\phi\left(\frac{x-\mu}{\sigma}\right)$

i) $P(x > 40)$ ii) $P(20 \leq x \leq 40)$

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given that $P(-\infty < t \le 2) = 0.9772$

where t denotes std normal variable

Given $\mu = 30$ $\sigma^2 = 25 \Rightarrow \sigma = 5$.

$$P(x>40) = 1 - P(X \le 40)$$

$$= 1 - \phi\left(\frac{40-30}{5}\right)$$

where $\phi(k) = P(t \le k)$

$$= 1 - \phi(2)$$

$$= 1 - 0.9772$$

$$\boxed{P(X>40) = 0.0228 \cdot \text{(ans)}}$$

$\phi(-2) = 1 - \phi(2)$

$P(x \le 40) - P(x \le 20)$

$$P(20 \le x \le 40) = \phi(2) - (1 - \phi(2))$$

$$= 2\phi(2) - 1$$

$$= 2 \times 0.9772 - 1$$

$$=$$

$$P(20 \le x \le 40) = P(X \le 40) - P(X \le 20)$$

$$= \phi\left(\frac{40-30}{}\right) - \phi\left(\frac{20-30}{}\right)$$

$P(20 \le x \le 40) = \ldots$

$$= \phi\left(\frac{40-30}{5}\right) - \phi\left(\frac{20-30}{5}\right)$$

$$= \phi(2) - \phi(-2)$$

$$= \phi(2) - (1 - \phi(2))$$

$$= 2\phi(2) - 1$$

$$= 2 \times 0.9772 - 1$$

$$= \underline{0.9544 \text{ (ans)}}$$

Find: i) $P(-31 < X < 67)$

ii) $P(X < 67 / X > 18)$

$$\int_{-\infty}^{1.96} (2\pi)^{-1/2} e^{-t^2/2}\, dt = 0.975002$$