Simultaneous system of Differential Equations

Consider 2 economic variables $y_1 y_2$, that are dependent.

$\therefore$ $\dot{y}_1 = f(y_1 y_2)$. and $\dot{y}_2 = f(y_1, y_2) \Rightarrow$ Find $y_1(t)$ & $y_2(t)$.

Linear system of diff eqns:

$$\left.\begin{aligned} \dot{y}_1 &= a_{11} y_1 + a_{12} y_2 + b_1 \text{ ------ } (i) \\ \dot{y}_2 &= a_{21} y_1 + a_{22} y_2 + b_2 \text{ ----- } (ii) \end{aligned}\right\} \rightarrow \text{Solve } y_1(t), y_2(t).$$

Matrix form: $\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$\boxed{\dot{y} = Ay + b} \rightarrow$. b: Long run equillibrium values of y_1, y_2

Finding complementary fn: $(b=0)$

$$\left.\begin{aligned} \Rightarrow \dot{y}_1 &= a_{11} y_1 + a_{12} y_2 \\ \Rightarrow \dot{y}_2 &= a_{21} y_1 + a_{22} y_2 \end{aligned}\right\} \rightarrow \text{Solve } y_1^c, y_2^c.$$

Finding particular integral (Put: $\dot{y}_1 = \dot{y}_2 = 0$).

$$\left.\begin{aligned} \Rightarrow 0 &= a_{11} y_1 + a_{12} y_2 + b_1 \\ 0 &= a_{21} y_1 + a_{22} y_2 + b_2 \end{aligned}\right\} \rightarrow \text{solve } y_1^p, y_2^p.$$

Complete soln: $\left.\begin{aligned} y_1(t) &= y_1^c + y_1^p. \\ y_2(t) &= y_1^c + y_2^p \end{aligned}\right\} \rightarrow$ Time paths of y_1 & y_2.

Q. Find the time path of y_1, y_2: $\left.\begin{aligned} \dot{y}_1 &= y_1 - 3y_2 - 5 \\ \dot{y}_2 &= \tfrac{1}{4} y_1 + 3y_2 - 5 \end{aligned}\right\}$ ---- $(*)$.

Given: $y_1(0) = 1$, $y_2(0) = 3$.

(I) Find complementary fn: $\left.\begin{aligned} \dot{y}_1 &= y_1 - 3y_2 \\ \dot{y} &= \tfrac{1}{4} y_1 + 3y_1 \end{aligned}\right\}$ ---- (i)

…any fn: $\left.\begin{aligned} \dot{y}_1 &= y_1 - 3y_2 \\ \dot{y}_2 &= \frac{1}{4}y_1 + 3y_2 \end{aligned}\right\}$ ---- (i)

$$\dot{y}_1 = y_1 - 3y_2 .$$

Diff: $\ddot{y}_1 = \dot{y}_1 - 3\dot{y}_2 = \dot{y}_1 - 3(0.25\, y_1 + 3y_2)$

$$\ddot{y}_1 = \dot{y}_1 - 0.75\, y_1 - 9\, y_2 .$$

$$y_2^c = \frac{y_1^c - \dot{y}_1^c}{3}$$

Eqn (i): $\dot{y}_1 = y_1 - 3y_2 \Rightarrow 3y_2 = y_1 - \dot{y}_1 \Rightarrow \boxed{y_2 = \dfrac{y_1 - \dot{y}_1}{3}}$

$$\ddot{y}_1 = \dot{y}_1 - 0.75\, y_1 - \cancel{9}^{\,3}\left(\frac{y_1 - \dot{y}_1}{\cancel{3}}\right)$$

$$\ddot{y}_1 = \dot{y}_1 - 0.75\, y_1 - 3y_1 + 3\dot{y}_1$$

Hom eqn of order 2 in y_1

$$\ddot{y}_1 = 4\dot{y}_1 - 3.75\, y_1 \Rightarrow \ddot{y}_1 - 4\dot{y}_1 + 3.75\, y_1 = 0$$

Ae^{rt} be trial $\Rightarrow r^2 - 4r + 3.75 = 0$

$$r = \frac{4 \pm \sqrt{16 - 4(3.75)}}{2} = \frac{5}{2}, \frac{3}{2}$$

$$y_1^c = A_1 e^{5/2\, t} + A_2 e^{3/2\, t}$$

Now: $\dot{y}_2 = \frac{1}{4}y_1 + 3y_2 \longrightarrow \dot{y}_2 - 3y_2 = \frac{1}{4}y_1$

$\Rightarrow y_1 = 4(\dot{y}_2 - 3y_2)$

Diff: $\ddot{y}_2 = \frac{1}{4}\dot{y}_1 + 3\dot{y}_2$

$$\ddot{y}_2 = \frac{1}{4}(y_1 - 3y_2) + 3\dot{y}_2$$

$$\ddot{y}_2 = \frac{1}{4}y_1 - \frac{3}{4}y_2 + 3\dot{y}_2$$

$$\ddot{y}_2 = \frac{1}{\cancel{4}}\cancel{4}(\dot{y}_2 - 3y_2) - \frac{3}{4}y_2 + 3\dot{y}_2$$

$$\ddot{y}_2 = \dot{y}_2 - 3y_2 - \frac{3}{4}y_2 + 3\dot{y}_2 = 4\dot{y}_2 - \frac{15}{4}y_2$$

$\ddot{y}_2 - 4\dot{y}_2 + \frac{15}{4}y_2 = 0 \rightarrow$ Diff eqn in y_2

$$y_2^c = -\frac{1}{2}A_1 e^{5/2\, t} + \frac{1}{6}A_2 e^{3/2\, t}$$

For particular integral: $\dot{y}_1 = \dot{y}_2 = 0$.

$$\left.\begin{aligned} y_1 - 3y_2 - 5 &= 0 \\ \tfrac{1}{4} y_1 + 3y_2 - 5 &= 0 \end{aligned}\right\} \Rightarrow y_1^p = 8, \quad y_2^p = 1.$$

$$\left.\begin{aligned} y_1(t) &= A_1 e^{5/2 t} + A_2 e^{3/2 t} + 8 \\ y_2(t) &= -\tfrac{1}{2} A_1 e^{5/2 t} + \tfrac{1}{6} A_2 e^{3/2 t} + 1 \end{aligned}\right\} \quad y_1(0) = 1, \quad y_2(0) = 3.$$

$$t=0: \left.\begin{aligned} y_1(0) &= A_1 + A_2 + 8 \Rightarrow 1 = A_1 + A_2 + 8 \\ y_2(0) &= -\tfrac{1}{2} A_1 + \tfrac{1}{6} A_2 + 1 \Rightarrow 3 = -\tfrac{1}{2} A_1 + \tfrac{1}{6} A_2 + 1 \end{aligned}\right\} \rightarrow \text{Solve } A_1, A_2.$$

Q. $r_1, r_2 = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i = h \pm v i$: $h = -1/2$, $v = \sqrt{3}/2$.

$$y_1^c = e^{ht}\left[A_3 \sin vt + A_4 \cos vt\right]$$

$$= e^{-1/2 t}\left[A_3 \sin\left(\tfrac{\sqrt{3}}{2} t\right) + A_4 \cos\left(\tfrac{\sqrt{3}}{2} t\right)\right]$$

$A_3 \rightarrow (A_1 - A_2) i$ $\qquad$ $A_4 \rightarrow (A_1 + A_2)$

$A_1 = p + qi$

$A_2 = p - qi$

$A_1 - A_2 = 2qi$

$(A_1 - A_2) i = -2q$

Direct Method: $\dot{y} = Ay + b$

Q. Given: $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\quad A = \begin{bmatrix} 2 & -5 \\ 2 & -4 \end{bmatrix}$ & system: $\dot{y} = Ay$.

Find time path of y_1 & y_2.

$y = k e^{rt}$ $\qquad\qquad$ $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} =$

$$\dot{y} = Ay \; \text{--- (i)}$$

From (i): $r k e^{rt} = A k e^{rt} \Rightarrow r e^{rt} = A e^{rt}$.

$$\Rightarrow [A - rI] k = 0.$$

Eigen-value: $|A - rI| = 0 = \begin{vmatrix} 2-r & -5 \\ 2 & -4-r \end{vmatrix} = 0$.

$$\Rightarrow -(2-r)(4+r)+10 = 0.$$

$$\Rightarrow (2-r)(4+r) = -10.$$

$$\Rightarrow 8+2r-4r-r^2 = -10.$$

$$\Rightarrow 8-2r-r^2 = -10.$$

$$r = -1 \pm \sqrt{17}\, i.$$