Q. Let $X_1, X_2, \ldots, X_n$ be a r.s from $U(\theta_1, \theta_2)$. Find the MLE of θ_1, θ_2.

$X \sim U(\theta_1, \theta_2)$ p.d.f $f(x) = \dfrac{1}{(\theta_2 - \theta_1)}$, $\theta_1 \le x \le \theta_2$.

(i) $L(\theta_1, \theta_2) = \prod_{i=1}^{n} f_{\theta_1, \theta_2}(x_i) = \prod_{i=1}^{n} \dfrac{1}{(\theta_2 - \theta_1)} = \dfrac{1}{(\theta_2 - \theta_1)^n}$

Obj: Max $L(\theta_1, \theta_2)$. $L(\theta_1, \theta_2)$ will be maximized if θ_2 is as small as possible and θ_1 is as large as possible.

R.S: $\theta_1 \le X_1, X_2, \ldots, X_n \le \theta_2$

Construct the ordered sample: $\theta_1 \le x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)} \le \theta_2$

$\hat{\theta}_{1, MLE} = x_{(n)}$ $\quad \hat{\theta}_{2, MLE} = x_{(1)}$.

Q. Suppose $X_1, X_2, \ldots, X_n$ be a r.s from $U(-\theta, \theta)$. Find MLE of θ.

Method of Moments

Let $X_1, X_2, \ldots, X_n$ be a r.s from $f_\theta(x)$, θ = unknown population parameter.

Population	Sample
(i) rth order moment about $x = a$: $= E(X-a)^r$, $r = 0, 1, 2, \ldots$	(i) rth order moment about $x = a$ $:= \frac{1}{n}\sum_{i=1}^{n}(x_i - a)^r$, $r = 0, 1, \ldots$
(ii) If $a = 0$, we call it rth order raw moment: $\mu_r' = E(X^r)$	(ii) If $a = 0$, rth order raw moment: $m_r' = \frac{1}{n}\sum_{i=1}^{n} x_i^r$.
(iii) If $a = \mu$, we call it rth order central moment: $\mu_r = E(X-\mu)^r$.	(iii) If $a = \bar{x}$, rth order central moment: $m_r = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^r$

$\therefore$ As per method of moments: $\boxed{\mu_r' = m_r'}$ $\Rightarrow$ using these eqns, find estimates of unknown population parameters.

Q. Let $X_1, X_2, \ldots, X_n$ be a r.s from $Exp(\theta)$. Find MoM estimate for θ.

$X_1, X_2, \ldots, X_n$, $\quad f(x) = \theta e^{-\theta x}$, $x \ge 0$.

$\therefore$ Consider first order raw moments for sample & popln.

$m_1' = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$

$\mu_1' = E(X) = \int_0^\infty x f(x)\, dx = \int_0^\infty x\, \theta\, e^{-\theta x}\, dx$

$$\mu_1' = E(X) = \int_0^\infty x f(x)\,dx = \int_0^\infty x\,\theta e^{-\theta x}\,dx$$

Let $\theta x = t \Rightarrow x = \left(\frac{t}{\theta}\right)$

$\theta\,dx = dt \Rightarrow dx = \frac{dt}{\theta}$

$x = 0,\ t = 0$

$x \to \infty,\ t \to \infty$

$$= \int_0^\infty \left(\frac{t}{\theta}\right) \theta\, e^{-t} \frac{dt}{\theta}$$

$$= \frac{1}{\theta}\int_0^\infty t^1 e^{-t}\,dt \quad \left[\text{Gamma Function}\right]$$

$$= \frac{1}{\theta}\int_0^\infty t^{2-1} e^{-t}\,dt = \Gamma 2/\theta$$

$$= 1/\theta.$$

Gamma Integral:

$$\Gamma p = \int_0^\infty e^{-x} x^{(p-1)}\,dx.$$

If p is an integer: $\Gamma p = (p-1)!$.

If p is not an integer: $\Gamma p = (p-1)\,\Gamma(p-1)$

$\Gamma 1/2 = \sqrt{\pi}$.

Using MoM:

$\mu_1' = m_1'$

$\frac{1}{\theta} = \bar{x}$

$\hat{\theta}_{MoM} = \frac{1}{\bar{x}}$

Q. Let $X_1, X_2, \dots, X_n$ be a r.s from $N(\mu, \sigma^2)$. Find the MoM estimate for μ, σ.

$m_1' = \mu_1'$ and $m_2' = \mu_2'$.

For sample: $m_1' = \frac{1}{n}\sum_{i=1}^n x_i = \bar{x}$.

$m_2' = \frac{1}{n}\sum_{i=1}^n x_i^2$

$E(x) = \mu$

$Var(x) = \sigma^2$

$E(x^2) - [E(x)]^2 = \sigma^2$

$E(x^2) - \mu^2 = \sigma^2$

$E(x^2) = \sigma^2 + \mu^2$.

For popln: $\mu_1' = E(X) = \mu$.

$\mu_2' = E(x^2) = \sigma^2 + \mu^2$.

From MoM estimation:

$\mu_1' = m_1'$

$\hat{\mu}_{MoM} = \bar{x}$

$\mu_2' = m_2'$.

$\sigma^2 + \mu^2 = \frac{1}{n}\sum_{i=1}^n x_i^2$

$\hat{\sigma}^2_{MoM} = \frac{1}{n}\sum x_i^2 - \hat{\mu}^2_{MoM}$

$= \frac{1}{n}\sum x_i^2 - \bar{x}^2 = \frac{1}{n}\sum (x_i - \bar{x})^2$.

$\therefore \hat{\sigma}_{MoM} = \sqrt{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2}$

Note: If we have a popln with k unknown parameters $(\theta_1, \theta_2, \dots, \theta_k)$. The MoM estimation will

$$\left.\begin{aligned} \mu_1' &= m_1' \\ \mu_2' &= m_2' \\ &\cdots \\ \mu_k' &= m_k' \end{aligned}\right\}$$

usings 'k' equations, solve for $\theta_1, \theta_2, \dots, \theta_k$.

i.e solving these give us: $\hat{\theta}_{1,MoM}, \hat{\theta}_{2,MoM}, \dots, \hat{\theta}_{k,MoM}$.