02 March 2024 06:32 PM

Weak Law of Large Numbers

(WLLN)

Let $X_1, X_2 \cdots X_n$ be a sequence of random variables and $\mu_1, \mu_2, \cdots \mu_n$ be their respective expectations and let

$$B_n = Var(X_1 + X_2 + \cdots + X_n) < \infty$$

Then $$P\left\{\left|\frac{X_1+X_2+\cdots+X_n}{n} - \frac{\mu_1+\mu_2+\cdots+\mu_n}{n}\right| < \varepsilon\right\} \geq 1-\eta$$

for all $n > n_0$, ε & η are small +ve no.s

provided $$\lim_{n\to\infty} \frac{B_n}{n^2} \to 0$$

Using Chebychev's inequality, to the random variable $\frac{X_1+X_2+\cdots+X_n}{n}$, we get for any $\varepsilon > 0$,

$$P\left\{\left|\underbrace{\left(\frac{X_1+X_2+\cdots+X_n}{n}\right)}_{} - \underbrace{\left[E\left(\frac{X_1+X_2+\cdots+X_n}{n}\right)\right]}_{}\right| < \varepsilon\right\} \geq 1 - \frac{B_n}{\ }$$

$$\geq 1 - \frac{B_n}{n^2 \varepsilon^2},$$

$$\left[\text{since } var\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) = \frac{1}{n^2} \left(Var(x_1 + x_2 + \cdots + x_n)\right) = \frac{B_n}{n^2}\right]$$

$$\rightarrow P\left\{\left|\frac{x_1 + x_2 + \cdots + x_n}{n} - \frac{\mu_1 + \mu_2 + \cdots + \mu_n}{n}\right| < \varepsilon\right\} \geq 1 - \frac{B_n}{n^2 \varepsilon^2}$$

As, we assume $n \to \infty$, $\frac{B_n}{n^2 \varepsilon^2} \to 0$ and since η is a small +ve no.

then we have the following inequality

ie, $\frac{B_n}{n^2 \varepsilon^2} < \eta$ will

hold for $n > n_0$

and consequently we shall have,

$$P\left\{\left|\frac{x_1 + x_2 + \cdots + x_n}{n} - \frac{\bar{\mu}_1 + \bar{\mu}_2 + \cdots + \bar{\mu}_n}{n}\right| < \varepsilon\right\} \geq 1 - \eta$$

Am of expectn

for all $n > n_0, (\varepsilon, \eta)$

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With the probability approaching unity, we may expect that the arithmetic mean of values actually assumed by 'n' random variables will differ from AM of their expectations by less than any given no., however small, provided the no. of variables can be taken sufficiently large and provided the condition $\frac{B_n}{n^2} \to 0$ as $n \to \infty$ is fulfilled.

Remarks : ① WLLN can also be stated as

$$\bar{X}_n \xrightarrow{P} \bar{\mu}_n$$

provided $\frac{B_n}{n^2} \to 0$ as $n \to \infty$.

② For the existence of the law we assume the following conditions:

(i) $E(x_i)$ exists.

(ii) $B_n = Var(x_1 + x_2 + \cdots + x_n)$ exists

* (iii) $\frac{B_n}{n^2} \to 0$ as $n \to \infty$

If the variables are uniformly bounded then the condition $\lim_{n \to \infty} \frac{B_n}{n^2} \to 0$ is necessary as well as sufficient condition for WLLN to hold.

Proof : Let $\xi_i = X_i - a_i$ $\Rightarrow$ { (deviation of X_i from it's mean) }

Proof : $\xi_i = x_i - a \Rightarrow \{$(deviation of x_i from it's mean)$\}$

where $E(x_i) = a_i$

then $E(\xi_i) = 0$ (for $i = 1, 2, \ldots, n$)

Since x_i's are uniformly bounded, there exists a positive number $c < \infty$ such that $|\xi_i| < c$

$$|U_n| \le n\varepsilon$$

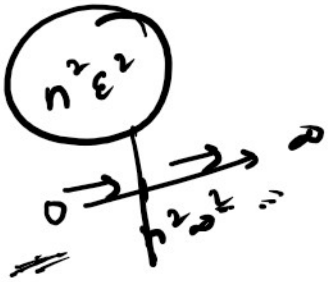
If $p = P\left[\,|\xi_1 + \xi_2 + \cdots + \xi_n| \le n\varepsilon\,\right]$

then $1 - p = P\left[\,|\xi_1 + \xi_2 + \cdots + \xi_n| > n\varepsilon\,\right]$

Let $U_n = \xi_1 + \xi_2 + \cdots + \xi_n$,

then $E(U_n) = \sum E(\xi_i) = 0$

and $\mathrm{var}(U_n) = E(U_n^2) = B_n$



$\Rightarrow B_n = \int_0^\infty U_n^2 \, dF(U_n)$ where $F(U_n)$ is d.f of U_n.

$$= \int_{U_n^2 \le n^2\varepsilon^2} U_n^2 \, dF + \int_{U_n^2 > n^2\varepsilon^2} U_n^2 \, dF.$$

$$B_n \le n^2\varepsilon^2 \int_{|U_n| \le n\varepsilon} dF + n^2 c^2 \int_{|U_n| > n\varepsilon} dF$$

$$B_n \leq n^2\varepsilon^2 \cdot p + n^2c^2(1-p)$$

$$\therefore \frac{B_n}{n^2} \leq \varepsilon^2 p + c^2(1-p)$$

If the law of large numbers holds,

$$1-p = P\left[\left|\xi_1 + \xi_2 + \dots + \xi_n\right| > n\varepsilon\right] \to 0$$
as $n \to \infty$

Hence as $n \to \infty$, $(1-p) \to 0$

and $\frac{B_n}{n^2} < \varepsilon^2 p + c^2\delta$, δ being arbitrarily small positive number

Hence $\frac{B_n}{n^2} \to 0$ as $n \to \infty$.

Bernoulli's Law of Large Numbers:

Let there be n trials of an event, each trial resulting in a success or failure.

If X is the no. of successes in 'n' trials with constant probability p of success for each trial, then $E(X) = np$ and $Var(X) = \underline{npq}$

where $q = 1-p$. The variable X/n represents the proportion of success or relative represents the proportion of success or relative frequency of success and

$$E(X/n) = \frac{1}{n} E(X) = \frac{np}{n} = p$$

$$Var(X/n) = \frac{1}{n^2} var(X) = \frac{npq}{n^2} = \frac{pq}{n}$$

Thm $P\left\{\left|\frac{X}{n} - p\right| < \varepsilon\right\} \to 1$ as $n \to \infty$

$\Rightarrow P\left\{\left|\frac{X}{n} - p\right| \geq \varepsilon\right\} \to 0$ as $n \to 0$

This means that (X/n) converges to probability p as $n \to \infty$.

Khinchin's Theorem:

If X_i's are identically and independently distributed random variables, the only condition necessary for the law of large no.s to hold is that $E(X_i)$; $i = 1, 2, \ldots$ should exist.

Q. A symmetric die is thrown 600 times. Find the lower bound for the probability of getting 80 to 120 sixes.

we have $n = 600$

Let S = no. of success

p = probability of success $= 1/6$

$q = 1 - p = 5/6$

$$E(S) = n \cdot p = 600 \times \frac{1}{6} = 100$$

$$V(S) = npq = 600 \times \frac{1}{6} \times \frac{5}{6} = 100 \times \frac{5}{6} = \frac{500}{6}$$

using chebyshev's inequality, we get,

$$P\left[\,|S - E(S)| < k\sigma\,\right] \geq 1 - \frac{1}{k^2}$$

$\sigma^2 = \frac{500}{6}$

$\sigma = \sqrt{500/6}$

$$P\left[\,|S - 100| < k\sqrt{500/6}\,\right] \geq 1 - \frac{1}{k^2}$$

$S - 100 < k\sqrt{500/6}$

$S < 100 + k\sqrt{500/6}$

$-S + 100 < k\sqrt{500/6}$

$S > 100 - k\sqrt{500/6}$

$$P\left[100 - k\sqrt{500/6} < S < 100 + k\sqrt{500/6}\right] \geq 1 - \frac{1}{k^2}$$

$$P\left[80 \leq S \leq 120\right] \geq 1 - \frac{1}{400 \times \frac{6}{500}}$$

$k = \dfrac{20}{\sqrt{500/6}}$

$1 - \frac{5}{24}$

$\frac{19}{24}$ ans

(2) Use chebyshev's inequality to determine how many times a fair coin must be tossed in order that the probability will be at least 0.90 that the ratio of observed no. of heads to the no. of tosses [lies between] 0.4 and 0.6

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