22 JUNE PG

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PG 22jun 2023 Que...

30. *The dimension of the vector space of all symmetric matrices* $A=(a_{ij})$ *of order* $n\times n (n\geq 2)$ *with real entries,* $a_{11}=0$ *and trace zero is*

1. $(n^2+n-4)/2$ 2. $(n^2-n+4)/2$ 3. $(n^2+n-3)/2$ 4. $(n^2-n+3)/2$

31. *Let N be the vector space of all real polynomials of degree at most 3. Define* $S: N \to N$ *by* $S(p(x)) = p(x+1)$, $p \in N$. *Then the matrix of S in the basis* $\{1, x, x^2, x^3\}$ *considered as column vectors is given by:*

1. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ *2.* $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *3.* $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix}$ *4.* $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

32. *Which of the following matrices are positive definite?*

1. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ ✓ ✗ 2. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ ✓ 3. $\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$ 4. $\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$ ✗

+ve definite → matrix is symmetric + All principal diagonal minors > 0.

(I) $A_{11} = 2 > 0$ $A_{22} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0$

(II) $B_{11} > 0$ $B_{22} < 0$

(III) $C_{11} > 0$ $C_{22} > 0$

(IV) $D_{11} = 0$ $D_{22} = -16 < 0$

dim = n - k = 0 = Eigen value of A

$A^2 = \lambda \cdot A$ $\quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times 2}$

$\rho(A) = 1 = k < n = 2$

Eigen value of A are 2, 0

33. Let A be a non-zero linear transformation on a real vector space V of dimension n. Let the subspace $V_0 \subset V$ be the image of V under A. Let $k = \dim V_0 < n$ and suppose that for some $\lambda \in \mathbb{R}$, $A^2 = \lambda A$. Then

1. $\lambda = 1$ ✗
2. $\det A = |\lambda|^n$ ✗
3. λ is the only eigenvalue of A ✗
4. there is a nontrivial subspace $V_1 \subset V$ such that $Ax = 0$ for all $x \in V_1$ ✓

$\det A = 0 \neq 2^2$

@ $\lambda = 2$ not the only E.V of A

0 is also E.V of A

34. *Let* C *be an* $n \times n$ *real matrix. Let* W *be the vector space spanned by* $\{I, C, C^2, \ldots, C^{2n}\}$. *The dimension of the vector space* W *is*

1. $2n$
2. *atmost* n
3. n^2
4. *atmost* $2n$

35. *Let V_1, V_2 be subspaces of a vector space V. Which of the following is/are necessarily a subspace of V?*

1. $V_1 \cap V_2$
2. $V_1 \cup V_2$
3. $V_1 + V_2 = \{x + y : x \in V_1, y \in V_2\}$
4. $V_1/V_2 = \{x \in V_1 \text{ and } x \notin V_2\}$

Easy ✓
medium
Hard

N is non zero nilpotent matrix. → never diagonalizable.

⇒ N is not similar to a diagonal matrix

36. Let N be a non-zero 3×3 matrix with the property $N^2 = O$. Which of the following is/are true?

1. N is not similar to a diagonal matrix. ✓
2. N is similar to a diagonal matrix. ✗
3. N has one non-zero eigenvector. ✓
4. N has three linearly independent eigenvectors. ✗

→ for >1 this does not hold

All E.v. = 0

$NX = 0 \cdot X$ ⇒ N has one non-zero eigenvector

So, option (3)

37. *Let n be a positive integer and let $M_n(\mathbb{R})$ denote the space of all $n \times n$ real matrices. If $T: M_n(\mathbb{R}) \to M_n(\mathbb{R})$ is a linear transformation such that $T(A)=0$ whenever $A \in M_n(\mathbb{R})$ is symmetric or skew-symmetric, then the rank of T is*

1. $\frac{n(n+1)}{2}$
2. $\frac{n(n-1)}{2}$
3. n
4. 0

38. *Let S: $\mathbb{R}^3 \to \mathbb{R}^4$ and T: $\mathbb{R}^4 \to \mathbb{R}^3$ be linear transformations such that $T \circ S$ is the identity map of $\mathbb{R}^3$. Then*

1. $S \circ T$ *is the identity map of* $\mathbb{R}^4$
2. $S \circ T$ *is one–one, but not onto.*
3. $S \circ T$ *is onto, but not one-one.*
4. $S \circ T$ *is neither one–one nor onto.*

39. *Let V be a 3-dimensional vector space over the field $F_3 = \mathbb{Z}/3\mathbb{Z}$ of 3 elements. The number of distinct 1- dimensional subspaces of V is*

1. 13
2. 26
3. 9
4. 15

40. *Let V be the inner product space consisting of linear polynomials, p: [0,1] $\to \mathbb{R}$ (i.e., V consists of polynomials p of the form $p(x) = ax + b$; $a , b \in \mathbb{R}$), with the inner product defined by*

$\langle p,q \rangle = \int_0^1 p(x)q(x)\,dx$ *for $p,q \in V$. An orthonormal basis of V is*

1. $\{1, x\}$
2. $\{1, x\sqrt{3}\}$
3. $\{1, (2x-1)\sqrt{3}\}$
4. $\left\{1, x-\frac{1}{2}\right\}$

medium

PUTNAM 2023

41. Let $f(x)$ be the minimal polynomial of the 4×4 matrix $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Then the rank of the 4 × 4 matrix $f(A)$ is .

1. 0   2. 1   3. 2   4. 4

Every matrix satisfies on minimal polynomial

$f(A)$ = zero matrix = 0

Rank $(f(A)) = 0$

(42.) Let a, b, c be positive real numbers such that $b^2 + c^2 < a < 1$. Consider the 3×3 matrix $A = \begin{bmatrix} 1 & b & c \\ b & a & 0 \\ c & 0 & 1 \end{bmatrix}$

1. All the eigenvalues of A are negative real numbers.
2. All the eigenvalues of A are positive real numbers.
3. A can have a positive as well as a negative eigenvalue.
4. Eigenvalues of A can be non-real complex numbers.

$A_{11} = 1 > 0$

$A_{22} = \begin{vmatrix} 1 & b \\ b & a \end{vmatrix} = a - b^2 > 0$ $\quad (0 < b^2 < b^2 + c^2 < a$, $a - b^2 > 0)$

$A_{33} = \begin{vmatrix} 1 & b & c \\ b & a & 0 \\ c & 0 & 1 \end{vmatrix} = a - b^2 - ac^2 > c^2 - ac^2 = c^2(1-a) > 0$

$\because b^2 + c^2 < a \quad a - b^2 > c^2$

$0 < a < 1$

$1 - a > 0 \quad c > 0$

$A_{33} > 0$

All principle minors > 0

matrix is symmetric

hence +ve definite

All Eigen Real

43. *The system of equations $x + y + z = 1$, $2x + 3y - z = 5$, $x + 2y - kz = 4$, where $k \in \mathbb{R}$, has an infinite number of solutions for*

1. $k = 0$
2. $k = 1$
3. $k = 2$
4. $k = 3$

$$\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 2 & 3 & -1 & : & 5 \\ 1 & 2 & -k & : & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & -3 & : & 3 \\ 0 & 1 & -k-1 & : & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & -3 & : & 3 \\ 0 & 0 & -k+2 & : & 0 \end{bmatrix}$$

Rank of coeff matrix = Rank of Augmented matrix

∞ no. soln

$-k + 2 = 0$

$k = 2$

44. *Let n be an integer, $n \geq 3$ and let $u_1, u_2, \ldots, u_n$ be n linearly independent elements in a vector space over $\mathbb{R}$. Set $u_0 = 0$ and $u_{n+1} = u_1$. Define $v_i = u_i + u_{i+1}$ and $w_i = u_{i-1} + u_i$ for $i = 1, 2, \ldots, n$. Then*

1. $v_1, v_2, \ldots, v_n$ *are linearly independent, if* $n = 2010$.
2. $v_1, v_2, \ldots, v_n$ *are linearly independent, if* $n = 2011$.
3. $w_1, w_2, \ldots, w_n$ *are linearly independent, if* $n = 2010$.
4. $w_1, w_2, \ldots, w_n$ *are linearly independent, if* $n = 2011$.

45. *Let V and W be finite-dimensional vector spaces over* $\mathbb{R}$ *and let* $T_1: V \to V$ *and* $T_2: W \to W$ *be linear transformations whose minimal polynomials are given by* $f_1(x) = x^3+x^2+x+1$ *and* $f_2(x) = x^4 - x^2 - 2$. *Let* $T: V \oplus W \to V \oplus W$ *be the linear transformation defined by* $T((v, w)) = (T_1(v), T_2(w))$ *for* $(v,w) \in V \oplus W$ *and let* $f(x)$ *be the minimal polynomial of T. Then*

1. $\deg f(x) = 7$
2. $\deg f(x) = 5$
3. nullity $(T) = 1$
4. nullity $(T) = 0$

46. *Let a, b, c, d* $\in \mathbb{R}$ *and let* $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ *be the linear transformation defined by* $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$ *for* $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. *Let* $S: \mathbb{C} \rightarrow \mathbb{C}$ *be the corresponding map defined by* $S(x + iy) = (ax + by) + i(cx + dy)$ *for* $x, y \in \mathbb{R}$. *Then*

1. S is always $\mathbb{C}$ – linear, that is $S(z_1 + z_2) = S(z_1) + S(z_2)$ for all $z_1, z_2 \in \mathbb{C}$ and $S(\alpha z) = \alpha S(z)$ for all $\alpha \in \mathbb{C}$ and $z \in \mathbb{C}$.
2. S is $\mathbb{C}$ – linear if $b = -c$ and $d = a$.
3. S is $\mathbb{C}$ – linear only if $b = -c$ and $d = a$.
4. S is $\mathbb{C}$ – linear if and only if T is the identity transformation.

47. Let $A = [a_{ij}]$ be an $n \times n$ complex matrix and let A^* denote the conjugate transpose of A. Which of the following statements are necessarily true?

1. If A is invertible, then $tr(A^*A) \neq 0$, i.e., the trace of A^*A is non zero.
2. If $tr(A^*A) \neq 0$, then A is invertible.
3. If $|tr(A^*A)| < n^2$, then $|a_{ij}| < 1$ for some i,j.
4. If $tr(A^*A) = 0$, then A is the zero matrix.

48. *Let n be a positive integer and V be an (n + 1)-dimensional vector space over* $\mathbb{R}$. *If* $\{e_1, e_2, \ldots, e_{n+1}\}$ *is a basis of V and* $T: V \to V$ *is the linear transformation satisfying* $T(e_i) = e_{i+1}$ *for* $i = 1, 2, \ldots, n$ *and* $T(e_{n+1}) = 0$. *Then*

1. *trace of T is non-zero.*
2. *rank of T is n.*
3. *nullity of T is 1*
4. $T^n = T \circ T \circ \ldots \circ T$ *(n times) is the zero map.*

49. Let A and B be $n \times n$ real matrices such that $AB = BA = O$ and $A + B$ is invertible. Which of the following are always true?

1. rank (A) = rank (B)
2. rank (A) + rank (B) = n.
3. nullity (A) + nullity (B) = n.
4. $A - B$ is invertible.

50. Let n be an integer ≥ 2 and let $M_n(\mathbb{R})$ denote the vector space of $n \times n$ real matrices. Let $B \in M_n(\mathbb{R})$ be an orthogonal matrix and let B^t denote the transpose of B. Consider $W_n = \{B^t AB : A \in M_n(\mathbb{R})\}$. Which of

50. *Let n be an integer* ≥ 2 *and let* $M_n(\mathbb{R})$ *denote the vector space of* $n \times n$ *real matrices. Let* $B \in M_n(\mathbb{R})$ *be an orthogonal matrix and let* B^t *denote the transpose of B. Consider* $W_B = \{B^t AB : A \in M_n(\mathbb{R})\}$. *Which of the following are necessarily true?*

1. W_B *is the subspace of* $M_n(\mathbb{R})$ *and* $\dim W_B \leq \text{rank}(B)$.
2. W_B *is the subspace of* $M_n(\mathbb{R})$ *and* $\dim W_B = \text{rank}(B)\,\text{rank}(B^t)$.
3. $W_B = M_n(\mathbb{R})$.
4. W_B *is not a subspace of* $M_n(\mathbb{R})$.

51. *Let A be a 5×5 skew-symmetric matrix with entries in $\mathbb{R}$ and B be the 5×5 symmetric matrix whose $(i,j)^{th}$ entry is the binomial coefficient $\binom{i}{j}$ for $1 \le i \le j \le 5$. Consider the 10×10 matrix, given in block form by $C = \begin{pmatrix} A & A+B \\ 0 & B \end{pmatrix}$. Then*

1. *det C =1 or -1*
2. *det C =0*
3. *trace of C is 0.*
4. *trace of C is 5*

52. *Suppose A is a 3 × 3 symmetric matrix such that* $[x, y, 1] A \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = xy - 1$. *Let p be the number of positive eigenvalues of A and let* $q = rank(A) - p$. *Then*

1. $p = 1$
2. $p = 2$
3. $q = 2$
4. $q = 1$

53. Let $A = \begin{bmatrix} 1 & 3 & 5 & a & 13 \\ 0 & 1 & 7 & 9 & b \\ 0 & 0 & 1 & 11 & 15 \end{bmatrix}$, *where* $a, b \in \mathbb{R}$. *Choose the correct statement.*

1. *There exist values of a and b for which the columns of A are linearly independent.*
2. *There exist values of a and b for which Ax=0 has x=0 as the only solution.*
3. *For all values of a and b, the rows of A span a 3-dimensional subspace of* $\mathbb{R}^5$.
4. *There exist values of a and b for which rank (A)=2.*

54. *Consider* $\mathbb{R}^3$ *with the standard inner product. Let W be the subspace of* $\mathbb{R}^3$ *spanned by (1,0,-1). Which of the following is a basis for the orthogonal complement of W?*

1. $\{(1,0,1),(0,1,0)\}$
2. $\{(1,2,1),(0,1,1)\}$
3. $\{(2,1,2),(4,2,4)\}$
4. $\{(2,-1,2),(1,3,1),(-1,-1,-1)\}$

55. *A linear transformation T rotates each vector in* $\mathbb{R}^2$ *clockwise through* 90^0. *The matrix T relative to the standard ordered basis* $\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)$ *is*

1. $\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$ 2. $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ 3. $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ 4. $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

56. *Let T:* $\mathbb{R}^n \rightarrow \mathbb{R}^n$ *be a linear transformation. Which of the following statements implies that T is bijective?*

1. *Nullity (T) =n*
2. *Rank(T) = Nullity (T) =n*
3. *Rank (T) + Nullity (T) =n*
4. *Rank(T) – Nullity (T) =n*

57. Let $A \in M_{10}(\mathbb{C})$, the vector space of 10×10 matrices with entries in $\mathbb{C}$. Let W_A be the subspace of $M_{10}(\mathbb{C})$ spanned by $\{A^n \mid n \geq 0\}$. Choose the correct statements.

1. For any A, $\dim(W_A) \leq 10$
2. For any A, $\dim(W_A) < 10$
3. For some A, $10 < \dim(W_A) < 100$
4. For some A, $\dim(W_A) = 100$

58. Let A be a complex 3×3 matrix with $A^3 = -I$. Which of the following statements are correct?

1. A has three distinct eigenvalues
2. A is diagonalizable over $\mathbb{C}$
3. A is triangularizable over $\mathbb{C}$
4. A is non-singular

59. *Consider the quadratic forms q and p given by* $q(x, y, z, w) = x^2 + y^2 + z^2 + bw^2$ *and* $p(x, y, z, w) = x^2 + y^2 + czw$. *Which of the following statements are true?*

1. *p and q are equivalent over* $\mathbb{C}$ *if b and c are non-zero complex numbers.*
2. *p and q are equivalent over* $\mathbb{R}$ *if b and c are non-zero real numbers.*
3. *p and q are equivalent over* $\mathbb{R}$ *if b and c are non-zero real numbers with b negative.*
4. *p and q are NOT equivalent over* $\mathbb{R}$ *if c=0*

60. *A linear operator T on a complex vector space V has characteristic polynomial* $x^3(x-5)^2$ *and minimal polynomial* $x^2(x-5)$. *Choose all correct options.*

1. *The Jordan form of T is uniquely determined by the given information*
2. *There are exactly 2 Jordan blocks in the Jordan decomposition of T*
3. *The operator induced by T on the quotient space V/Ker(T-5I) is nilpotent, where I is the identity operator*
4. *The operator induced by T on the quotient space V/Ker(T) is a scalar multiple of the identity operator*

61. *Let S denote the set of all primes p such that the following matrix is invertible when considered as a matrix with entries in* $\mathbb{Z}/p\mathbb{Z}$.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ -2 & 0 & 2 \end{pmatrix}$$

. Which of the following statements are true?

1. *S contains all the prime numbers*
2. *S contains all the prime numbers greater than 10*
3. *S contains all the prime numbers other than 2 and 5*
4. *S contains all the odd prime numbers.*

62. *For a fixed positive integer $n \geq 3$, let A be the $n \times n$ matrix defined as $A = I - \frac{1}{n}J$, where I is the identity matrix and J is the $n \times n$ matrix with all entries equal to 1. Which of the following statements is NOT true?*

1. $A^k = A$ for every positive integer k.
2. Trace $(A) = n-1$
3. Rank(A)+Rank$(I - A) = n$.
4. A is invertible.

63. *Let A be a* 5×4 *matrix with real entries such that* $A\underline{x} = \underline{0}$ *if and only if* $\underline{x} = \underline{0}$, *where* $\underline{x}$ *is a* 4×1 *vector and* $\underline{0}$ *is a null vector. Then, the rank of A is*

1. 4 *2. 5* *3. 2* *4. 1*

64. *Consider the following row vectors*

$\alpha_1 = (1,1,0,1,0,0)$, $\alpha_2 = (1,1,0,0,1,0)$

$\alpha_3 = (1,1,0,0,0,1)$, $\alpha_4 = (1,0,1,1,0,0)$

$\alpha_5 = (1,0,1,0,1,0)$, $\alpha_6 = (1,0,1,0,0,1)$

The dimension of the vector space spanned by these row vectors is

1. 6 *2. 5* *3. 4* *4. 3*

65. *Let* $A_{n\times n} = ((a_{ij})), n \geq 3,$ *where* $a_{ij} = \left(b_i^2 - b_j^2\right),\ i, j = 1,2,\ldots,n$ *for some distinct real numbers* $b_1, b_2, \ldots, b_n$. *Then det(A) is*

1. $\prod_{i<j}(b_i - b_j)$ 2. $\prod_{i<j}(b_i + b_j)$ 3. 0 4. 1

66. *Let A be an* $n \times n$ *matrix with real entries. Which of the following is correct?*

1. *If* $A^2=O$, *then A is diagonalizable over complex numbers.*
2. *If* $A^2=I$, *then A is diagonalizable over real numbers*
3. *If* $A^2=A$, *then A is diagonalizable only over complex numbers.*
4. *The only matrix of size n satisfying the characteristic polynomial of A is A.*

67. *Let A be a* 4×4 *invertible real matrix. Which of the following is NOT necessarily true?*

1. *The rows of A form a basis of* $\mathbb{R}^4$.
2. *Null space of A contains only the 0 vector.*
3. *A has 4 distinct eigenvalues.*
4. *Image of the linear transformation* $x \mapsto Ax$ *on* $\mathbb{R}^4$ *is* $\mathbb{R}^4$.

68. *Let* $\{v_1,\dots,v_n\}$ *be a linearly independent subset of a vector space V, where* $n \geq 4$. *Set* $w_{ij} = v_i - v_j$. *Let W be the span of* $\{w_{ij} \mid 1 \leq i, j \leq n\}$. *Then*

1. $\{w_{ij} \mid 1 \leq i < j \leq n\}$ *spans W.*
2. $\{w_{ij} \mid 1 \leq i < j \leq n\}$ *is a linearly independent subset of W.*
3. $\{w_{ij} \mid 1 \leq i \leq n-1, j = i+1\}$ *spans W.*
4. *dim* $W = n$

69. *For any real square matrix M, let $\lambda^{+}(M)$ be the number of positive eigenvalues of M counting multiplicities. Let A be an $n \times n$ real symmetric matrix and Q be an $n \times n$ real invertible matrix. Then*

1. Rank A = Rank $Q^{T}AQ$
2. Rank A = Rank $Q^{-1}AQ$
3. $\lambda^{+}(A) = \lambda^{+}(Q^{T}AQ)$
4. $\lambda^{+}(A) = \lambda^{+}(Q^{-1}AQ)$

70. *Let T_1, T_2 be two linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis of $\mathbb{R}^n$. Suppose that $T_1(x_i) \neq 0$ for every $i = 1, 2, \ldots, n$ and that $x_i \perp \text{Ker } T_2$ for every $i = 1, 2, \ldots, n$. Which of the following is/are necessarily true?*

1. T_1 *is invertible*
2. T_2 *is invertible*
3. *Both* T_1, T_2 *are invertible*
4. *Neither* T_1 *nor* T_2 *is invertible*

71. Let $S: \mathbb{R}^n \to \mathbb{R}^n$ be given by $v \mapsto \alpha v$ for a fixed $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $B = \{v_1, \ldots, v_n\}$ is a set of linearly independent eigenvectors of T. Then

1. The matrix of T with respect to B is diagonal.
2. The matrix of $T - S$ with respect to B is diagonal.
3. The matrix of T with respect to B is not necessarily diagonal, but upper triangular.
4. The matrix of T with respect to B is diagonal but the matrix of $(T-S)$ with respect to B is not diagonal.

72. *For an* $n\times n$ *real matrix A,* $\lambda\in\mathbb{R}$ *and a nonzero vector* $v\in\mathbb{R}^n$*, suppose that* $(A-\lambda I)^k v=0$ *for some positive integer k. Let I be the* $n\times n$ *identity matrix. Then which of the following is/are always true?*

1. $(A-\lambda I)^{k+r} v=0$ *for all positive integers r.*
2. $(A-\lambda I)^{k-1} v=0$
3. $(A-\lambda I)$ *is not injective*
4. λ *is an eigenvalue of A*

73. *Let y be a non-zero vector in an inner product space V. Then which of the following are subspaces of V?*

1. $\{x \in V \mid \langle x, y \rangle = 0\}$.
2. $\{x \in V \mid \langle x, y \rangle = 1\}$.
3. $\{x \in V \mid \langle x, z \rangle = 0$ *for all z such that* $\langle z, y \rangle = 0\}$.
4. $\{x \in V \mid \langle x, z \rangle = 1$ *for all z such that* $\langle z, y \rangle = 1\}$.

74. *Let A be a* 5×5 *matrix with real entries such that the sum of the entries in each row of A is 1. Then the sum of all the entries in* A^3 *is*

1. 3 *2. 15* *3. 5* *4. 125*

75. Let J denote a 101×101 matrix with all the entries equal to 1 and let I denote the identity matrix of order 101. Then the determinant of $J-I$ is

1. 101
2. 1
3. 0
4. 100

76. Let $M_{m\times n}(\mathbb{R})$ be the set of all $m\times n$ matrices with real entries. Which of the following statements is correct?

1. There exists $A\in M_{2\times 5}(\mathbb{R})$ such that the dimension of the null space of A is 2.
2. There exists $A\in M_{2\times 5}(\mathbb{R})$ such that the dimension of the null space of A is 0.
3. There exist $A\in M_{2\times 5}(\mathbb{R})$ and $B\in M_{5\times 2}(\mathbb{R})$ such that AB is the 2×2 identity matrix.
4. There exists $A\in M_{2\times 5}(\mathbb{R})$ whose null space is $\{(x_1, x_2, x_3, x_4, x_5)\in \mathbb{R}^5 : x_1=x_2, x_3=x_4=x_5\}$

77. *For the matrix A as given below, which of them satisfy $A^6 = I$?*

1. $A = \begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ 0 & -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$

3. $A = \begin{pmatrix} \cos\frac{\pi}{6} & 0 & \sin\frac{\pi}{6} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{6} & 0 & \cos\frac{\pi}{6} \end{pmatrix}$

4. $A = \begin{pmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} & 0 \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

78. *Let V denote a vector space over a field F and with a basis B={$e_1, e_2, \ldots, e_n$}. Let* $x_1, x_2, \ldots, x_n \in F$. *Let* $C = \{x_1e_1, x_1e_1 + x_2e_2, \ldots, x_1e_1 + x_2e_2 + \ldots + x_ne_n\}$. *Then*

1. *C is linearly independent set implies that* $x_i \neq 0$ *for every i=1,2,…,n.*
2. $x_i \neq 0$ *for every i=1,2,…,n implies that C is linearly independent set.*
3. *The linear span of C is V implies that* $x_i \neq 0$ *for every i=1,2,…,n.*
4. $x_i \neq 0$ *for every i=1,2,…,n implies that the linear span of C is V.*

79. *Let V denote the vector space of all polynomials over $\mathbb{R}$ of degree less than or equal to n. Which of the following defines a norm on V?*

1. $\| p \|^2 = | p(1) |^2 + ... + | p(n+1) |^2, \; p \in V$
2. $\| p \| = \sup_{t \in [0,1]} | p(t) |, \; p \in V$
3. $\| p \| = \int_0^1 | p(t) | \, dt, \; p \in V$
4. $\| p \| = \sup_{t \in [0,1]} | p'(t) |, \; p \in V$

80. Let u, v, w be vectors in an inner- product space V, satisfying $||u||=||v||=||w||=2$ and $<u,v>=0$, $<u,w> = 1$, $<v,w>= -1$. Then which of the following are true?

1. $||w+v-u||=2\sqrt{2}$.
2. $\left\{\frac{1}{2}u, \frac{1}{2}v\right\}$ forms an orthonormal basis of a two dimensional subspace of V.
3. w and $4u-w$ are orthogonal to each other.
4. u, v, w are necessarily linearly independent.

81. Let A be a 4×4 matrix over $\mathbb{C}$ such that $rank(A)=2$ and $A^3 = A^2 \neq 0$. Suppose that A is not diagonalizable. Then

1. One of the Jordan blocks of the Jordan canonical form of A is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
2. $A^2 = A \neq 0$.
3. There exists a vector v such that $Av \neq 0$ but $A^2 v = 0$.
4. The characteristic polynomial of A is $x^4 - x^3$.

82. *Let* $\varphi: \mathbb{R}^2 \to \mathbb{C}$ *be the map defined by* $\varphi(x, y) = z$, *where* $z=x+iy$. *Let* $f: \mathbb{C} \to \mathbb{C}$ *be the function* $f(z)=z^2$ *and* $F = \varphi^{-1} f \varphi$. *Which of the following are correct?*

1. *The linear transformation* $T(x, y) = 2\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ *represents the derivative of F at (x,y).*
2. *The linear transformation* $T(x, y) = 2\begin{pmatrix} x & y \\ y & x \end{pmatrix}$ *represents the derivative of F at (x,y).*
3. *The linear transformation* $T(z)=2z$ *represents the derivative of f at* $z \in \mathbb{C}$.
4. *The linear transformation* $T(z)=2z$ *represents the derivative of f only at 0.*

83. Let V be the vector space of polynomials over $\mathbb{R}$ of degree less than or equal to n. For $p(x) = a_0 + a_1x + \ldots + a_nx^n$ in V, define a linear transformation $T : V \to V$ by $(Tp)(x) = a_0 - a_1x + a_2x^2 - \ldots + (-1)^n a_nx^n$. Then which of the following are correct?

1. T is one-to-one
2. T is onto
3. T is invertible
4. $\det T = 0$

84. *Consider a homogeneous system of linear equations* $Ax=0$, *where* A *is an* $m \times n$ *real matrix and* $n > m$. *Then which of the following statements are always true?*

1. $Ax=0$ *has a solution.*
2. $Ax=0$ *has no non-zero solution.*
3. $Ax=0$ *has a non-zero solution.*
4. *Dimension of the space of all solutions is at least* $n-m$.

85. *Let A, B be $n \times n$ matrices such that $BA + B^2 = I - BA^2$, where I is the $n \times n$ identity matrix. Which of the following is always true?*

1. A is nonsingular *2. B is nonsingular* *3. A+B is nonsingular* *4. AB is nonsingular*

86. *Which of the following matrices has the same row space as the matrix* $\begin{pmatrix} 4 & 8 & 4 \\ 3 & 6 & 1 \\ 2 & 4 & 0 \end{pmatrix}$?

1. $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *2.* $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *3.* $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *4.* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

87. *The determinant of the* $n \times n$ *permutation matrix* $\begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & \end{bmatrix}$

1. $(-1)^n$ 2. $(-1)^{\left[\frac{n}{2}\right]}$ 3. -1 4. 1

Here, [x] denotes the greatest integer not exceeding x.

88. *The determinant* $\begin{vmatrix} 1 & 1+x & 1+x+x^2 \\ 1 & 1+y & 1+y+y^2 \\ 1 & 1+z & 1+z+z^2 \end{vmatrix}$ *is equal to*

1. $(z-y)(z-x)(y-x)$
2. $(x-y)(x-z)(y-z)$
3. $(x-y)^2(y-z)^2(z-x)^2$
4. $(x^2-y^2)(y^2-z^2)(z^2-x^2)$

89. *Which of the following matrices is not diagonalizable over* $\mathbb{R}$?

1. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

90. *Let P be a 2×2 complex matrix such that P^*P is the identity matrix, where P^* is the conjugate transpose of P. Then the eigenvalues of P are*

1. real
2. complex conjugates of each other
3. reciprocals of each other
4. of modulus 1

91. *Let A be a real $n \times n$ orthogonal matrix, that is, $A^t A = AA^t = I_n$, the $n \times n$ identity matrix. Which of the following statements are necessarily true?*

1. $\langle Ax, Ay \rangle = \langle x, y \rangle \; \forall \, x, y \in \mathbb{R}^n$
2. *All eigenvalues of A are either +1 or -1.*
3. *The rows of A form an orthonormal basis of $\mathbb{R}^n$.*
4. *A is diagonalizable over $\mathbb{R}$.*

92. *Which of the following matrices have Jordan canonical form equal to* $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$?

1. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
2. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
3. $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
4. $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

93. *Let A be a* 3×4 *and b be a* 3×1 *matrix with integer entries. Suppose that the system* $Ax=b$ *has a complex solution. Then*

1. $Ax=b$ has an integer solution
2. $Ax=b$ has a rational solution
3. The set of real solutions to $Ax=0$ has a basis consisting of rational solutions.
4. If $b\neq0$, then A has positive rank.

94. *Let f be a non-zero symmetric bilinear form on $\mathbb{R}^3$. Suppose that there exist linear transformations $T_i : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2$ such that for all $\alpha, \beta \in \mathbb{R}^3$, $f(\alpha, \beta) = T_1(\alpha) T_2(\beta)$. Then*

1. *rank $f = 1$*
2. *$\dim \{\beta \in \mathbb{R}^3 : f(\alpha, \beta) = 0$ for all $\alpha \in \mathbb{R}^3\} = 2$*
3. *f is positive semi-definite or negative semi-definite.*
4. *$\{\alpha : f(\alpha, \alpha) = 0\}$ is a linear subspace of dimension 2*

95. *The matrix* $A = \begin{pmatrix} 5 & 9 & 8 \\ 1 & 8 & 2 \\ 9 & 1 & 0 \end{pmatrix}$ *satisfies*

1. *A is invertible and the inverse has all integer entries.*

2. det(A) is odd.
3. det(A) is divisible by 13.
4. det(A) has at least two prime divisors.

96. *Let A be* 5×5 *matrix and let B be obtained by changing one element of A. Let r and s be the ranks of A and B respectively. Which of the following statements is/are correct?*

1. $s \le r+1$ 2. $r-1 \le s$ 3. $s = r-1$ 4. $s \ne r$

97. Let $M_n(K)$ denote the space of all $n \times n$ matrices with entries in a field K. Fix a non-singular matrix $A = (A_{ij}) \in M_n(K)$, and consider the linear map $T : M_n(K) \to M_n(K)$ given by T(X)=AX. Then

1. trace (T)=n $\sum_{i=1}^{n} A_{ii}$
2. trace(T) $= \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}$
3. rank of T is n^2
4. T is non–singular

98. *For arbitrary subspaces U,V and W of a finite dimensional vector space, which of the following hold?*

1. $U \cap (V+W) \subset U \cap V + U \cap W$
2. $U \cap (V+W) \supset U \cap V + U \cap W$
3. $(U \cap V)+W \subset (U+W) \cap (V+W)$
4. $(U \cap V)+W \supset (U+W) \cap (V+W)$

99. *Let A be a* 4×7 *real matrix and B be a* 7×4 *real matrix such that* $AB = I_4$, *where* I_4 *is the* 4×4 *identity matrix. Which of the following is/are always true?*

1. *rank (A)=4*
2. *rank (B)=7*
3. *nullity (B)=0*
4. $BA=I_7$, *where* I_7 *is the* 7×7 *identity matrix*

100. *Let* $\mathbb{R}[x]$ *denote the vector space of all real polynomials. Let* $D: \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ *denote the map* $Df = \frac{df}{dx}, \forall f$. *Then,*

1. *D is one-one*
2. *D is onto*
3. *There exists* $E: \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ *so that* $D(E(f)) = f, \forall f$.
4. *There exists* $E: \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ *so that* $E(D(f)) = f, \forall f$.