17 June 2023 10:16

$i=\sqrt{-1}$ $\omega \to$ cube root of unity

$\omega^3=1$

$\boxed{\omega^2+\omega+1=0}$

$\boxed{i^2=-1}$ $i^3=-i$ $i^4=1$

$\omega^2=-\omega-1$

Let i be a root of the equation $x^2+1=0$ and let ω be a root of the equation $x^2+x+1=0$. Construct a polynomial

$$f(x)=a_0+a_1x+\cdots+a_nx^n$$

where $a_0, a_1, \ldots, a_n$ are all integers such that $f(i+\omega)=0$.

$$f(i+\omega)=a_0+a_1(i+\omega)+a_2(i+\omega)^2+\cdots+a_n(i+\omega)^n.$$

$$(i+\omega)^2=i^2+2i\omega+\omega^2=i^2+2i\omega-\omega-1=-1+2i\omega-\omega-1=\boxed{-2+2i\omega-\omega}$$

$$(i+\omega)^3=(i+\omega)(i+\omega)^2=(i+\omega)(-2+2i\omega-\omega)=-2i-i\omega+2i^2\omega-2\omega-\omega^2+2i\omega^2$$

$$=-2i-i\omega-2\omega-2\omega+\omega+1-2i(\omega+1)=\boxed{1-4i-3\omega-3i\omega}$$

$$(i+\omega)^4=(i+\omega)(i+\omega)^3=(i+\omega)(1-4i-3\omega-3i\omega)=i-4i^2-3i\omega-3i^2\omega+\omega-4i\omega$$
$$-3\omega^2-3i\omega^2=i+4-3i\omega+3\omega+\omega-4i\omega+3(\omega+1)+3i(\omega+1)$$
$$=\boxed{7+4i+7\omega-4i\omega}$$

$$0=f(i+\omega)=a_0+a_1(i+\omega)+a_2(i+\omega)^2+a_3(i+\omega)^3+a_4(i+\omega)^4$$

$$a_0+a_1(i+\omega)+a_2(-2+2i\omega-\omega)+a_3(1-4i-3\omega-3i\omega)+a_4(7+4i+7\omega-4i\omega)=0$$

$$(a_0-2a_2+a_3+7a_4)+(a_1-4a_3+4a_4)i+(a_1-a_2-3a_3+7a_4)\omega$$
$$+(2a_2-3a_3-4a_4)i\omega=0.$$

$$a_0-2a_2+a_3+7a_4=0.$$
$$a_1-4a_3+4a_4=0.$$
$$a_1-a_2-3a_3+7a_4=0$$
$$2a_2-3a_3-4a_4=0$$

4 equations, 5 unknowns a_0, a_1, a_2, a_3, a_4

$$a_0=2a_2-a_3-7a_4$$
$$=10a_4-2a_4-7a_4$$
$$\boxed{a_0=a_4}$$

$$\boxed{a_1=4a_3-4a_4}$$
$$\boxed{a_1=4a_4}$$

$$4a_3-4a_4-a_2-3a_3+7a_4=0.$$
$$\boxed{a_2=a_3+3a_4}\ \boxed{=5a_4}$$

$$2(a_3+3a_4)-3a_3-4a_4=0 \Rightarrow \boxed{a_3=2a_4}$$

$$\boxed{a_4=1}\quad a_0=1,\ a_1=4,\ a_2=5,\ a_3=2.$$

$$f(x)=1+4x+5x^2+2x^3+x^4.$$

Let a be a fixed real number. Consider the equation

$$(x+2)^2(x+7)^2 + a = 0,\ x \in \mathbb{R}$$

where $\mathbb{R}$ is the set of real numbers. For what values of a, will the equation have exactly one double root?

$\boxed{f(x) = g(x) + a}$ where $\boxed{g(x) = (x+2)^2(x+7)^2}$.

$g'(x) = 2(x+2)(x+7)^2 + 2(x+2)^2(x+7) = 2(x+2)(x+7)(2x+9)$

maxima / minima $\rightarrow$ $\boxed{x = -2, -7}$, $-\frac{9}{2}$.

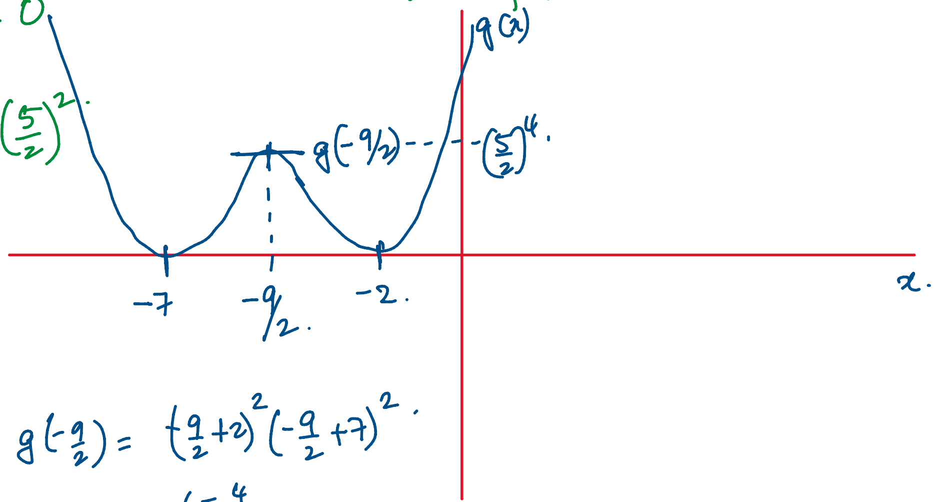
roots of $g(x)$ $\rightarrow$ minima.

$f(-\frac{9}{2}) = (\frac{5}{2})^2 + a = 0$

$\Downarrow$

double root

$a = -(\frac{5}{2})^2$.



$g(-\frac{9}{2}) = (-\frac{9}{2}+2)^2(-\frac{9}{2}+7)^2$.

$= (\frac{5}{2})^4$

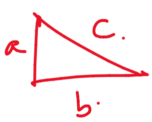
Consider a right-angled triangle with integer-valued sides $a < b < c$ where a, b, c are pairwise co-prime. Let $d = c - b$. Suppose d divides a. Then

$\Rightarrow d^2$ divides a^2.

$c = b + d$.

(a) Prove that $d \leq 2$

(b) Find all such triangles (i.e., all possible triplets a,b,c) with perimeter less than 100.



$a^2 + b^2 = c^2 \Rightarrow a^2 = c^2 - b^2 = (c+b)(c-b) = d(c+b)$

$a^2 = d(c+b) = d^2 k$.

$\boxed{c+b = dk}$

$\therefore$ d divides c+b

$c+b = b+d+b = 2b+d$.

$c = b+d$.

$d + 2b = kd$.

$\therefore$ d divides b.

$\therefore$ d divides b. $\quad c = b + d$. $\quad d + 2b = kd$.

$2b = (k-1)d$.

$\therefore$ d divides c.

let $p \geq 3$ be a prime factor of d.

$\Rightarrow$ p divides d $\Rightarrow$ $\boxed{p \text{ divides } b \Rightarrow p \text{ divides } c}$

p is a common factor of b & c

assumption is invalid

$\therefore$ $p = 2$. is the only prime factor of d.

$\therefore d = 1, 2$.

$d=1$ $\quad c = b+1 \Rightarrow b^2+2b+1 = c^2 = a^2+b^2 \Rightarrow \boxed{a^2 = 2b+1} \Rightarrow$ a is odd

let $\boxed{a = 2k+1}$

$(2k+1)^2 = 2b+1 \Rightarrow 4k^2+4k+1 = 2b+1 \Rightarrow \boxed{b = 2k^2+2k}$

$\boxed{c = b+1 = 2k^2+2k+1}$

Perimeter < 100

$4k^2+6k+2 < 100$.

$2k^2+3k+1 < 50$.

$\boxed{1 \leq k \leq 4}$

4 Δs. $\quad (3,4,5), (5,12,13), (7,24,25), (9,40,41)$

$d=2$ $\quad c = b+2$.

$b^2+4b+4 = c^2 = a^2+b^2 \Rightarrow a^2 = 4b+4 \Rightarrow$ a is even. $\Rightarrow \boxed{a = 2k}$

$4k^2 = a^2 = 4b+4. \Rightarrow \boxed{b = k^2-1} \Rightarrow \boxed{c = k^2+1}$

$2k^2+2k < 100. \Rightarrow k^2+k < 50.$ $\quad \boxed{k \leq 6}$. $\quad (8,15,17)$, $(12,35,37)$

2 Δs

Let $x_1 = \prod_{r=1}^{5} \cos\frac{r\pi}{11}$ and $x_2 = \sum_{r=1}^{5} \cos\frac{r\pi}{11}$, then show that

$x_1 \cdot x_2 = \frac{1}{64}\left(\operatorname{cosec}\frac{\pi}{22} - 1\right)$, where Π denotes the continued product.

$\frac{1}{2}\left(\operatorname{cosec}\frac{\pi}{22} - 1\right)$

$= \frac{1}{2}\left[\frac{1}{\sin\frac{\pi}{22}} - 1\right]$

$$x_1 = \left(\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11}\right) \times \frac{2\sin\frac{\pi}{11}}{2\sin\frac{\pi}{11}}$$

$\sin(A+B) + \sin(A-B)$

$= 2\sin A\cos B$.

$$= \frac{1}{2 \times 2\sin\frac{\pi}{11}}\left[2\sin\frac{2\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11}\right] = \frac{1}{2 \times 4\sin\frac{\pi}{11}}\left[2\sin\frac{4\pi}{11}\cos\frac{4\pi}{11}\cos\frac{3\pi}{11}\cos\frac{5\pi}{11}\right]$$

$\sin 11\frac{\pi}{11} = \sin\pi = 0$.

$$= \frac{1}{2 \times 8\sin\frac{\pi}{11}}\left[2\sin\frac{8\pi}{11}\cos\frac{3\pi}{11}\cos\frac{5\pi}{11}\right] = \frac{1}{16\sin\frac{\pi}{11}}\left[2\sin\frac{8\pi}{11}\cos\frac{3\pi}{11}\right]\cos\frac{5\pi}{11}$$

$\sin\frac{10\pi}{11} = \sin\left(\pi - \frac{\pi}{11}\right)$

$= \sin\frac{\pi}{11}$

$$= \frac{1}{16\sin\frac{\pi}{11}}\left[\sin 11\frac{\pi}{11} + \sin\frac{5\pi}{11}\right]\cos\frac{5\pi}{11} = \frac{1}{2 \times 16\sin\frac{\pi}{11}} 2 \times \sin\frac{5\pi}{11}\cos\frac{5\pi}{11}$$

$$= \frac{1}{32\sin\frac{\pi}{11}}\sin\frac{10\pi}{11} = \frac{\sin\pi/11}{32\sin\frac{\pi}{11}} = \frac{1}{32}$$

$$x_2 = \cos\frac{\pi}{11} + \cos\frac{2\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{4\pi}{11} + \cos\frac{5\pi}{11} = \cos\frac{2\pi}{22} + \cos\frac{4\pi}{22} + \cos\frac{6\pi}{22}$$

$$+ \cos\frac{8\pi}{22} + \cos\frac{10\pi}{22}$$

$$\sin(A+B)+\sin(A-B) = 2\sin A\cos B$$

$$\sin(-\theta) = -\sin\theta$$

$$= \frac{1}{2\sin\frac{\pi}{22}}\left[2\sin\frac{\pi}{22}\cos\frac{2\pi}{22} + 2\sin\frac{\pi}{22}\cos\frac{4\pi}{22} + 2\sin\frac{\pi}{22}\cos\frac{6\pi}{22} + 2\sin\frac{\pi}{22}\cos\frac{8\pi}{22} + 2\sin\frac{\pi}{22}\cos\frac{10\pi}{22}\right]$$

$$= \frac{1}{2\sin\frac{\pi}{22}}\left[\cancel{\sin\frac{3\pi}{22}} - \sin\frac{\pi}{22} + \cancel{\sin\frac{5\pi}{22}} - \cancel{\sin\frac{3\pi}{22}} + \cancel{\sin\frac{7\pi}{22}} - \cancel{\sin\frac{5\pi}{22}} + \cancel{\sin\frac{9\pi}{22}} - \cancel{\sin\frac{7\pi}{22}} + \sin\frac{11\pi}{22} - \cancel{\sin\frac{9\pi}{22}}\right]$$

$$= \frac{1}{2\sin\frac{\pi}{22}}\left[\sin\frac{\pi}{2} - \sin\frac{\pi}{22}\right] = \frac{1}{2\sin\frac{\pi}{22}}\left[1 - \sin\frac{\pi}{22}\right]$$

$$= \frac{1}{2}\left[\operatorname{cosec}\frac{\pi}{22} - 1\right]$$