Convergence of Sequences

A seq $\{x_n\}$ has a finite limit 'l' $\Rightarrow \lim_{n\to\infty} x_n = l$.

Theorem 1: If limit of a seq $\{x_n\}$ exists, then it is unique.

Theorem 2: Convergent seq $\Rightarrow$ Bounded seq. Eg: $x_n = (-1)^n$
$\not\Leftarrow$

$$\{x_n\} = \{-1, 1, -1, 1, \ldots\}$$

Note: $\{x_n\} = \left\{\frac{1}{n^3}\right\} = 0$. $\therefore$ In general $\{x_n\} = \frac{1}{n^p}, p > 1$

$= \left\{1 + \frac{1}{n^3}\right\} \to 1$

seq will always be convergent.

Sandwich Theorem:

Consider 3 seq $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ s.t $x_n \le y_n \le z_n$ $\forall n > N$. Then if x_n and z_n are convergent then y_n would be convergent as well.

convergent

i.e If $\lim_{n\to\infty} x_n = l$ and $\lim_{n\to\infty} z_n = l$, then by sandwich Theorem: $\lim_{n\to\infty} y_n = l$.

Note: $\{x_n\}$ and $\{z_n\}$ needs to be constructed to conclude about convergence of $\{y_n\}$.

Q. $\{y_n\} = (3^n + 4^n)^{1/n}$. Check if x_n is convergent or not.

Now, $3^n / \quad 4^n < 3^n + 4^n < 4^n + 4^n$

$$4 < (3^n + 4^n)^{1/n} < 2^{1/n} \cdot 4.$$

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$$\swarrow \qquad \downarrow \qquad \searrow$$

$x_n \qquad\qquad\qquad z_n$

$$\lim_{n\to\infty} x_n = 4 \qquad\qquad \lim_{n\to\infty} z_n = \lim_{n\to\infty} 2^{1/n} \cdot 4 = 4 .$$

$$\lim_{n\to\infty} y_n = 4 .$$

Q. $$\{y_n\} = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+3}} + \cdots + \frac{1}{\sqrt{n^2+n}} .$$

$$y_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots\cdots + \frac{1}{\sqrt{n^2+n}}$$

$$< \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \cdots\cdots + \frac{1}{\sqrt{n^2+1}} = \frac{n}{\sqrt{n^2+1}} = z_n$$

$$y_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$$

$$> \frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + \cdots + \frac{1}{\sqrt{n^2+n}} = \frac{n}{\sqrt{n^2+n}} = x_n$$

$$\therefore \quad \frac{n}{\sqrt{n^2+n}} < y_n < \frac{n}{\sqrt{n^2+1}} .$$

$$\left.\begin{aligned} \lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = 1 . \\ \lim_{n\to\infty} z_n &= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1 \end{aligned}\right\} \Rightarrow \lim_{n\to\infty} y_n = 1$$

Q. $\{y_n\} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \cdots$ strictly increasing

$$y_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

$$\times \begin{cases} \frac{1}{n!} + \frac{1}{n!} + \cdots\cdots + \frac{1}{n!} \\ \frac{1}{1!} + \frac{1}{1!} + \cdots\cdots + \frac{1}{1!} = n \to \infty . \end{cases}$$

$$\frac{1}{2!} = \frac{1}{2\times 1}$$

$$\frac{1}{3!} = \frac{1}{3\times 2\times 1} < \frac{1}{2\times 2} = \frac{1}{2^2} = \frac{1}{2^{3-1}}$$

$$\frac{1}{4!} = \frac{1}{4\times 3\times 2\times 1} < \frac{1}{2\times 2\times 2} = \frac{1}{2^3} = \frac{1}{2^{4-1}}$$

In general: $\frac{1}{k!} < \frac{1}{2^{k-1}}$ $\forall k > 2$.

$$y_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\cdots + \frac{1}{n!}$$

$$< 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\cdots + \frac{1}{2^{n-1}}$$

[GP with $a = 1,\ r = \frac{1}{2}$, terms $= n$.

$$= \frac{1\left(1-\left(\frac{1}{2}\right)^n\right)}{\left(1-\frac{1}{2}\right)} = 2\left(1-\left(\frac{1}{2}\right)^n\right) = 2 - \frac{1}{2^{n-1}}$$

$$\lim_{n\to\infty} 2 - \frac{1}{2^{n-1}} = 2$$

HW.

Q. $x_{n+1} = \sqrt{2x_n}$. Check if $\{x_n\}$ is convergent.

$x_1 = \sqrt{2}$, $x_2 = \sqrt{2\sqrt{2}}$, $x_3 = \sqrt{2\sqrt{2\sqrt{2}}}$, ...

Note: (i) If seq $\{x_n\}$ is strictly increasing and bounded above, then $\{x_n\}$ is convergent.

(ii) If seq $\{x_n\}$ is strictly decreasing and bounded below, then $\{x_n\}$ is convergent.