Digression:

Normal Distributions and its associated extensions:-

If $X \sim N(\mu, \sigma^2)$: $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $-\infty < x < \infty$

$-\infty < \mu < \infty$

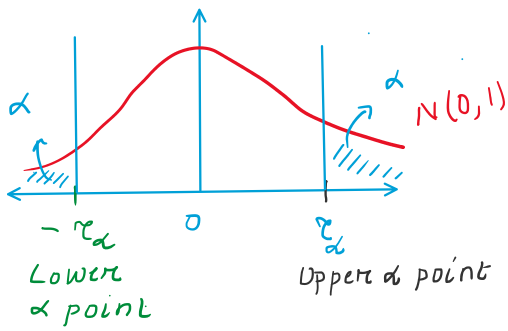
$\sigma > 0$, finite

(i) Standard Normal Distribution:

$$Z/\tau = \frac{X-\mu}{\sigma} \sim N(0,1)$$

$\Rightarrow$ Symmetric about zero.

$\Rightarrow$ $E(\tau) = 0$, $Var(\tau) = 1$

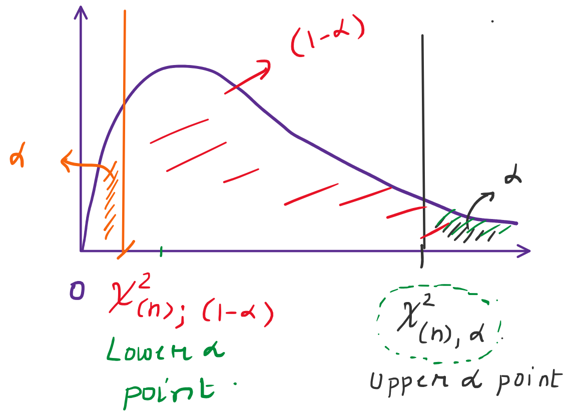


(ii) Chi-Square Distribution:

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$$

$$\Rightarrow \frac{X_i - \mu}{\sigma} \sim N(0,1) \quad \forall\, i = 1, 2, \ldots, n$$

$$Y = \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_{(n)} \qquad d.f = n.$$



$\Rightarrow$ Positively skewed.

Note: μ, σ = popln parameters.

If μ, σ are known, $Y = \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$

But if μ is unknown, replace it $\bar{X}$, $Y' = \sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2 \sim \chi^2_{(n-1)}$

d.f = No. of independent variables/

No. of variables that can take independent values.

= No. of variables − No. of restrictions. $= (n-1)$

$(Y' = (X_1 - \bar{X})^2 \; (X_2 - \bar{X})^2 \; \ldots \; (X_n - \bar{X})^2$

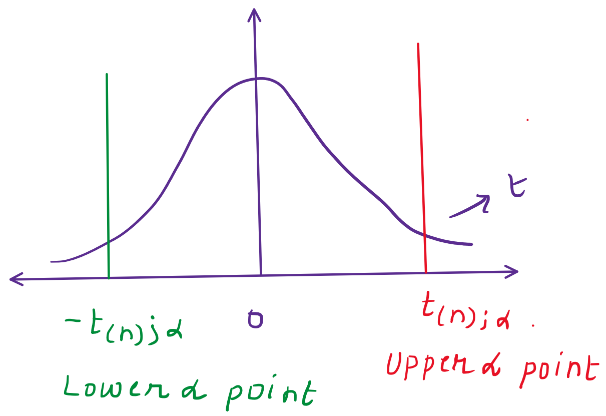
No of Restrictions = $(n-1)$

$$Y' = \left(\frac{X_1 - \bar{X}}{\sigma}\right)^2 + \left(\frac{X_2 - \bar{X}}{\sigma}\right)^2 + \cdots + \left(\frac{X_n - \bar{X}}{\sigma}\right)^2$$ ⇒ Adding 'n' variables

Restriction: $\sum_{i=1}^{n} (x_i - \bar{x}) = 0$.

(iii) t - distribution:

$$t = \frac{Z}{\sqrt{\chi^2_{(n)}/n}} \sim t_{(n)}$$



⇒ symmetric about zero.

⇒ Diff b/w Normal, t-distribution:

↳ t is more peaked (leptokurtic distribution)

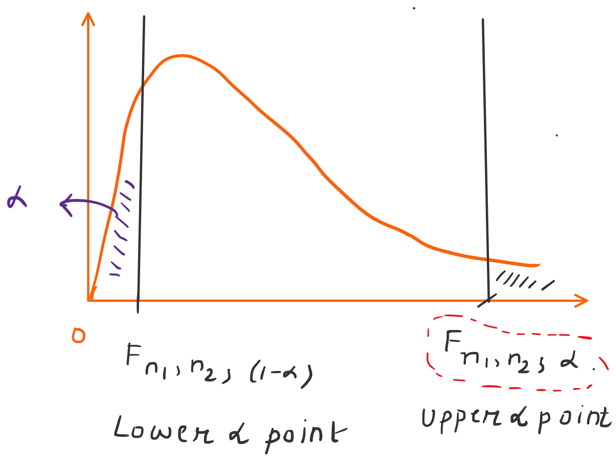
↳ Normal is less peaked (mesokurtic distribution)

(iv). F-distribution.

Consider two independent χ^2 variates:

$\chi^2_{(n_1)}$ with $df = n_1$ and $\chi^2_{(n_2)}$ with $df = n_2$

$$F = \frac{\chi^2_{(n_1)}/n_1}{\chi^2_{(n_2)}/n_2} \sim F_{n_1, n_2}$$



⇒ Positively skewed.

$$\Rightarrow \quad F' = \frac{\chi^2_{(n_2)}/n_2}{\chi^2_{(n_1)}/n_1} \sim F_{n_2, n_1}$$

Find the upper α point for $F'_{n_2, n_1; \alpha} = \dfrac{1}{F_{n_1, n_2; \alpha}}$.

II Testing for Population Variance:

To test: $H_0: \sigma^2 = \sigma_0^2$ vs $H_{1A}: \sigma^2 > \sigma_0^2$ } → one-tailed test

$\rightarrow H_{1B}: \sigma^2 < \sigma_0^2$

$H_{1C}: \sigma^2 \neq \sigma_0^2$ } → two-tailed test.

Case I: If μ is known.

$$T = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \overset{H_0}{\sim} \chi^2_{(n)}$$

(i) We will reject H_0 at α% L.O.S if $T_{obs} > \chi^2_{(n);\alpha}$.

(ii) We will reject H_0 at α% L.O.S if $T_{obs} < \chi^2_{n;(1-\alpha)}$

(iii) We will reject H_0 at α% L.O.S if $T_{obs} > \chi^2_{(n);\alpha/2}$

or $T_{obs} < \chi^2_{(n);(1-\alpha/2)}$

Case II: If μ is unknown.

$$T = \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma}\right)^2 \overset{H_0}{\sim} \chi^2_{(n-1)}$$

(i)

(ii)

(iii)

III Testing for Difference in population means.

Consider two independent populations and let us draw r.s of size n_1 & n_2 respectively from these populations.

r.s. 1 [from popln 1]: $X_{11}, X_{12}, \ldots, X_{1n_1}$.

r.s. 2 [from popln 2]: $X_{21}, X_{22}, \ldots, X_{2n_2}$.

To test: $H_0: \mu_1 = \mu_2 \Rightarrow \mu_1 - \mu_2 = 0$ vs $H_1 \ldots$

To test: $H_0: \mu_1 = \mu_2 \Rightarrow \boxed{\mu_1 - \mu_2 = 0}$ vs $H_{1A}: \mu_1 - \mu_2 > 0$.

$H_{1B}: \mu_1 - \mu_2 < 0$.

$H_{1C}: \mu_1 - \mu_2 \neq 0$.

Case I: σ_1^2, σ_2^2 are known.

$$\therefore \quad X_1 \sim N(\mu_1, \sigma_1^2) \Rightarrow \boxed{\bar{X}_1 \sim N\left(\mu_1, \frac{\sigma_1^2}{n_1}\right)}$$

$$X_2 \sim N(\mu_2, \sigma_2^2) \Rightarrow \boxed{\bar{X}_2 \sim N\left(\mu_2, \frac{\sigma_2^2}{n_2}\right)}$$

$$E(\bar{X}_1 - \bar{X}_2) = E(\bar{X}_1) - E(\bar{X}_2) = (\mu_1 - \mu_2)$$

$$Var(\bar{X}_1 - \bar{X}_2) = Var(\bar{X}_1) + Var(\bar{X}_2) - 2\underbrace{cov(\bar{X}_1, \bar{X}_2)}_{=0}$$

$$= Var(\bar{X}_1) + Var(\bar{X}_2)$$

$$= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

$$\therefore \quad (\bar{X}_1 - \bar{X}_2) \sim N\left[(\mu_1 - \mu_2), \left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)\right].$$

$$\therefore \quad T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \overset{H_0}{\sim} N(0,1) \quad \text{[Test-statistic]}$$

HW.

(i)

(ii)

(iii)