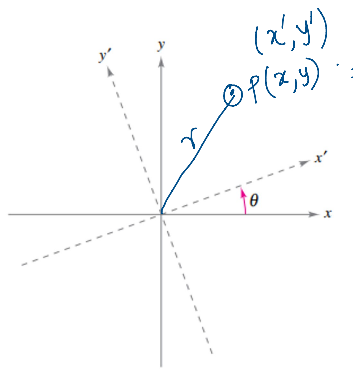
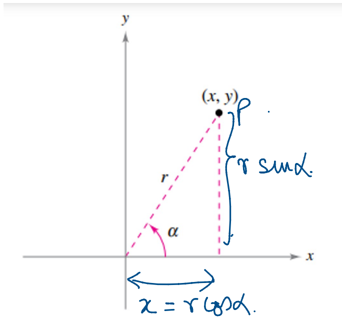
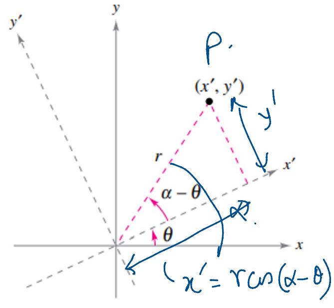
30 January 2024 14:59

Rotation and the General Second-Degree Equation





Original: $x = r\cos\alpha$
$y = r\sin\alpha$



Rotated: $x' = r\cos(\alpha-\theta)$
$y' = r\sin(\alpha-\theta)$

Relationship between (x, y) and (x', y')

$$x' = r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = \underbrace{\boxed{r\cos\alpha}}_{x}\cos\theta + \underbrace{\boxed{r\sin\alpha}}_{y}\sin\theta$$

$$x' = x\cos\theta + y\sin\theta \quad \text{— (1)}$$

$$y' = r(\sin\alpha\cos\theta - \cos\alpha\sin\theta) = \underbrace{\boxed{r\sin\alpha}}_{y}\cos\theta - \underbrace{\boxed{r\cos\alpha}}_{x}\sin\theta$$

$$y' = y\cos\theta - x\sin\theta \quad \text{— (2)}$$

from (1) → & $x'\cos\theta = x\cos^2\theta + y\sin\theta\cos\theta$

from (2) → $y'\sin\theta = y\cos\theta\sin\theta - x\sin^2\theta$

$$x'\cos\theta - y'\sin\theta = x \quad \text{— (3)}$$

from (1) → $x'\sin\theta = x\cos\theta\sin\theta + y\sin^2\theta$

from (2) → $y'\cos\theta = y\cos^2\theta - x\sin\theta\cos\theta$

$$x'\sin\theta + y'\cos\theta = y \quad \text{→ (4)}$$

Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

replace x with $x'\cos\theta - y'\sin\theta$
y with $x'\sin\theta + y'\cos\theta$

$$\frac{(x'\cos\theta - y'\sin\theta)^2}{a^2} + \frac{(x'\sin\theta + y'\cos\theta)^2}{b^2} = 1$$

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$$

replace x' with $x\cos\theta + y\sin\theta$
y' with $y\cos\theta - x\sin\theta$

$$\frac{(x\cos\theta + y\sin\theta)^2}{a^2} + \frac{(y\cos\theta - x\sin\theta)^2}{b^2} = 1$$

$$\frac{x^2\cos^2\theta}{a^2} + \frac{y^2\sin^2\theta}{a^2} + \frac{2\cos\theta\sin\theta\, xy}{a^2} + \frac{x^2\sin^2\theta}{b^2} + \frac{y^2\cos^2\theta}{b^2} - \frac{2\cos\theta\sin\theta\, xy}{b^2} = 1$$

$$\underbrace{\boxed{\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)}}_{A} x^2 + \underbrace{\boxed{\sin 2\theta\left(\frac{1}{a^2} - \frac{1}{b^2}\right)}}_{B} xy + \underbrace{\boxed{\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right)}}_{C} y^2 = 1$$

$$Ax^2 + Bxy + Cy^2 - 1 = 0$$

$$\boxed{B \neq 0 \Rightarrow \text{rotation}}$$

$$\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = A \quad -(1)$$

$$\sin 2\theta\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = B \quad -(2)$$

$$\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} = C \quad -(3)$$

$$(1) - (3) \Rightarrow \frac{\cos 2\theta}{a^2} - \frac{\cos 2\theta}{b^2} = A - C$$

$$\cos 2\theta\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = A - C \quad -(4)$$

$$(4) \div (2) \Rightarrow \cot 2\theta = \frac{A-C}{B} \checkmark$$

$$\tan 2\theta = \frac{B}{A-C}$$

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{B}{A-C}$$

$xy = 1 \rightarrow \boxed{Ax^2 + Bxy + Cy^2 + D = 0}$

$A = 0$

$B = 1$

$C = 0$

find θ

$$\cot 2\theta = \frac{A-C}{B} = 0$$

$$2\theta = 90^\circ \rightarrow \underline{\underline{\theta = 45^\circ}}$$

replace x, y in terms of x', y'

$$\left.\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= y'\cos\theta + x'\sin\theta \end{aligned}\right\} \theta = 45^\circ$$

$$x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} \qquad y = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}$$

$y = y \cos\theta + x \sin\theta$

$$x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} \qquad y = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}$$

$$xy = 1$$

$$\Rightarrow \left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) = 1 \quad \Rightarrow \frac{x'^2}{2} - \frac{y'^2}{2} = 1$$

$$\Rightarrow \frac{x'^2}{(\sqrt{2})^2} - \frac{y'^2}{(\sqrt{2})^2} = 1$$

standard form.

Summary

$$\boxed{Ax^2 + Bxy + Cy^2} + Dx + Ey + F = 0$$

Step 1 Calculate θ using $\cot 2\theta = \frac{A-C}{B}$.

Step 2 $\left\{ \begin{array}{l} x = x'\cos\theta - y'\sin\theta \\ y = y'\cos\theta + x'\sin\theta \end{array} \right\} \rightarrow$ put it back in given equation and simplify to identify the conic.

$$\frac{x'^2}{a^2} \pm \frac{y'^2}{b^2} = 1 \qquad x'^2 = 4ay' \,/\, y'^2 = 4ax$$

for parabola $\quad x^2 = 4ay$.

$$(x'\cos\theta - y'\sin\theta)^2 = 4a(y'\cos\theta + x'\sin\theta)$$

$$x'^2\cos^2\theta + y'^2\sin^2\theta - 2\cos\theta\sin\theta\, x'y' = 4ay'\cos\theta + 4ax'\sin\theta$$

$$(\cos^2\theta)\, x'^2 - (\sin 2\theta)\, x'y' + (\sin^2\theta)\, y'^2 - (4a\sin\theta)\, x' - (4a\cos\theta)\, y' = 0$$

for ellipse/ hyperbola $\quad \frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$

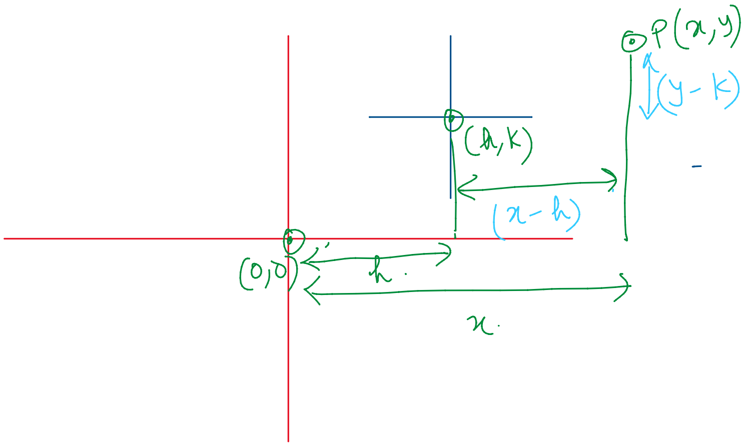
shift origin to (h, k)

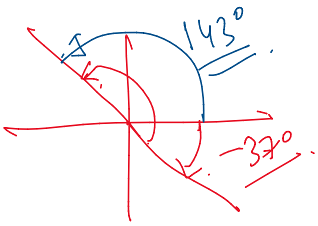
$$\frac{(x-h)^2}{a^2} \pm \frac{(y-k)^2}{b^2} = 1$$

rotate axis

replace x and y

Shift of origin





$$2x^2 + 3xy + 6y^2 + 5 = 0.$$

$A = 2 \quad B = 3 \quad C = 6,$

$$\frac{\cos 2\theta}{\sin 2\theta} = \cot 2\theta = \frac{A-C}{B} = \frac{-4}{3} \quad \frac{\cos -ve}{\sin +ve} \quad 2\theta = -37^\circ$$

$$2\theta = -37^\circ + 180^\circ = 143^\circ$$

$$\theta = \frac{143^\circ}{2} = 71.5^\circ$$